Postnikov–Stanley polynomials are Lorentzian
(arXiv:2412.02051)

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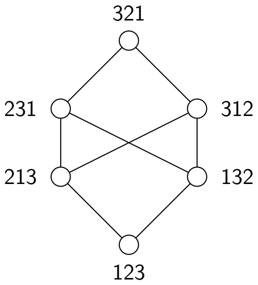
The (strong) Bruhat order on S_n

- S_n: permutations of $\{1, \ldots, n\}$
- $\mathrm{Inv}(u)$: the set of all inversions (a, b) of u such that $a < b$ and $u(a) > u(b)$
- $\ell(u)$: count of inversions in u
- t_{ab} swaps the numbers in positions a, b (not values a, b)
- Covering relation: $u \lessdot v$ if $v = ut_{ab}$ and $\ell(v) = \ell(u) + 1$
- Interval $[u, w]$: $\{v \mid u \leq v \leq w\}$



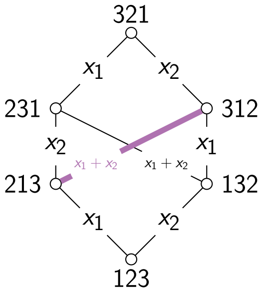
Edge weights

Definition

For $u \lessdot v$ and $v = ut_{ab}$, the *weight* $m(u \lessdot v)$ is $x_a + x_{a+1} + \cdots + x_{b-1}$.

Example

Since $312 = 213t_{13}$, we have $m(213 \lessdot 312) = x_1 + x_2$.



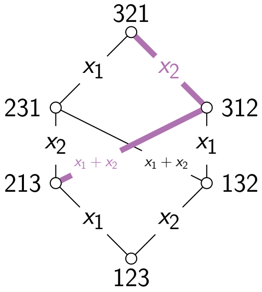
Chain weights

Definition

Let $u_0 \leq u_\ell$ and $C = (u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell)$ be a saturated chain of $[u_0, u_\ell]$. Define the *weight* $m_C(x)$ of the chain C by $\prod_{i=1}^{\ell} m(u_{i-1} \lessdot u_i)$.

Example

For $[213, 321]$, the weight of the saturated chain $213 \lessdot 312 \lessdot 321$ is $(x_1 + x_2) \cdot x_2$.



Postnikov–Stanley polynomials

Definition (Postnikov–Stanley '09)

For $u \leq w$ in the Bruhat order on S_n, the skew dual Schubert polynomial or *Postnikov–Stanley polynomial* D_u^w is defined by

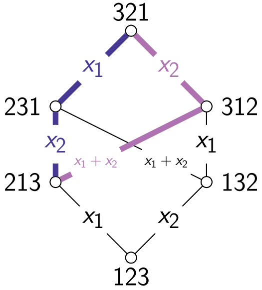
$$D_u^w = \frac{1}{(\ell(w)-\ell(u))!} \sum_{C: u=u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell = w} m_C(x).$$

Example

$D_{213}^{321} = \frac{1}{2!}(x_1 x_2 + (x_1 + x_2) \cdot x_2)$.

Definition (Bernstein–Gelfand–Gelfand '73)

D_{id}^w is called a *dual Schubert polynomial.*



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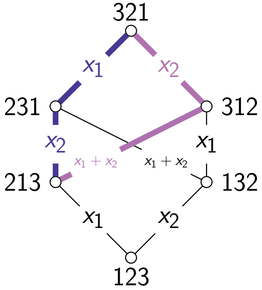

Dual Schubert polynomials

Let I be an indexing set, and let $\{f_a\}_{a\in I}$ be a collection of homogeneous polynomials so that, for each n, its projection to $\mathbb{C}[x_1, \dots, x_n]$ forms a basis for that ring after removing nonzero elements.

Definition

Collection $\{f_a\}$ is *dual* to collection $\{g_a\}$ if $\sum_a f_a(x_1, x_2, \dots)g_a(y_1, y_2, \dots) = e^{x_1y_1+x_2y_2+\cdots}$ where $x_i := y_i - y_{i+1}$.

Theorem (Postnikov–Stanley '09)

The Schubert polynomials $\{\mathfrak{S}_w\}_{w\in S_\infty}$ are dual to the polynomials $\{D_{\mathrm{id}}^w\}_{w\in S_\infty}$.

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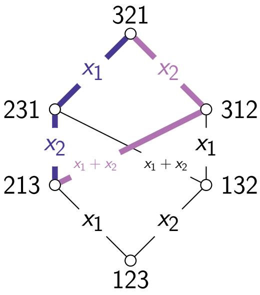
Saturated Newton polytope (SNP)

For a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0}$, let $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

Definition

The *support* $\mathrm{supp}(f)$ of $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha$ is the set of vectors α such that $c_\alpha \neq 0$. The *Newton polytope* $\mathrm{Newton}(f)$ is the convex hull of $\mathrm{supp}(f)$ in $\mathbb{R}^n$.



Example

$D^{321}_{213} = x_1 x_2 + \frac{1}{2} x_2^2 = x^{(1,1)} + \frac{1}{2} x^{(0,2)}$.
$\mathrm{Newton}(D^{321}_{213})$ is the segment from $(1,1)$ to $(0,2)$.

Definition (Monical–Tokcan–Yong '19)

The polynomial f has *saturated Newton polytope (SNP)* if $\mathrm{supp}(f)$ is the set of integer points in $\mathrm{Newton}(f)$.

SNP in algebraic combinatorics

Theorem (Rado '52)

Schur polynomials have SNP.

Theorem (Fink–Mézśaros–St. Dizier '18)

Key polynomials and Schubert polynomials have SNP.

Theorem (Monical–Tokcan–Yong '19)

Cycle index polynomials and symmetric Macdonald polynomials have SNP.

Theorem (Castillo–Cid Ruiz–Mohammadi–Montaño '19)

Double Schubert polynomials have SNP.

Theorem (Huh–Matherne–Mészáros–St. Dizier '19; An–T.–Zhang '24)

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Chain weights have SNP

Definition (Postnikov–Stanley '09)

$$D_u^w = \frac{1}{(\ell(w) - \ell(u))!} \sum_{C: u = u_0 \lessdot u_1 \lessdot \cdots \lessdot u_\ell = w} m_C(x).$$

Proposition (An–T.–Zhang '24)

Any product of linear factors in $x_1, \ldots, x_n$ with all coefficients nonnegative has SNP.

In particular, each chain weight $m_C(x)$ has SNP.

Single-chain Newton polytope (SCNP)

Definition (An–T.–Zhang '24)

D_u^w has *single-chain Newton polytope (SCNP)* if there exists a saturated chain C in the interval $[u, w]$ such that

$$\mathrm{supp}(m_C) = \mathrm{supp}(D_u^w).$$

We call such a C a *dominant chain* of the interval $[u, w]$.

Proposition (An–T.–Zhang '24)

If D_u^w has SCNP, then D_u^w has SNP.

Example and nonexample of SCNP

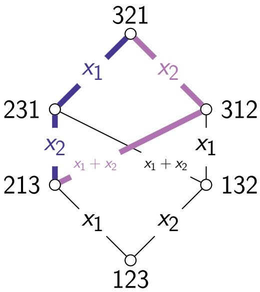
Example

$D^{321}_{213} = \frac{1}{2!}(x_1x_2 + (x_1 + x_2) \cdot x_2)$ has SCNP.

$$C := (213 \lessdot 312 \lessdot 321)$$

$$m_C = (x_1 + x_2) \cdot x_2$$

$$\mathrm{supp}(m_C) = \mathrm{supp}(D^{321}_{213})$$



Example

D^{4231}_{1324} does not have SCNP.

Dual Schubert polynomials have SCNP

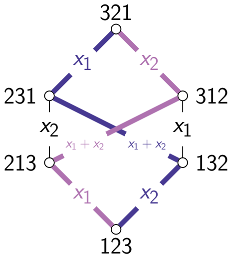
Definition (An–T.–Zhang '24)

A saturated chain $u = w_0 \lessdot w_1 \lessdot w_2 \lessdot \cdots \lessdot w_\ell = w$ is *greedy* if it satisfies the following for all $i \in [\ell]$: writing $w_{i-1} t_{ab} = w_i$ for $a < b$, there does not exist $w'_{i-1} \lessdot w_i$ with $w'_{i-1} \in [u, w]$ such that

$$w'_{i-1} t_{ab'} = w_i \text{ for } b' > b, \text{ or } \; w'_{i-1} t_{a'b} = w_i \text{ for } a' < a.$$

Example

In $[123, 321]$,
$123 \lessdot 132 \lessdot 231 \lessdot 321$ is greedy.
$123 \lessdot 213 \lessdot 312 \lessdot 321$ is also greedy.
$123 \lessdot 213 \lessdot 231 \lessdot 321$ is not greedy.



Dual Schubert polynomials have SCNP (2)

Definition (An–T.–Zhang '24)

The *global weight* $\mathrm{GW}(w)$ of $w \in S_n$ is

$$\mathrm{GW}(w) = \prod_{(a,b) \in \mathrm{Inv}(w)} (x_a + x_{a+1} + \cdots + x_{b-1}).$$

Example

$\mathrm{Inv}(231) = \{(1,3),(2,3)\}$, $\mathrm{GW}(231) = (x_1 + x_2) \cdot x_2$.

Theorem (An–T.–Zhang '24)

For all $w \in S_n$, the dual Schubert polynomial D^w has SCNP. Moreover, every greedy chain of $[\mathrm{id}, w]$ is a dominant chain of D^w, and

$$\mathrm{supp}(D^w) = \mathrm{supp}(\mathrm{GW}(w)) = \sum_{(a,b) \in \mathrm{Inv}(w)} \{e_a, e_{a+1}, \ldots, e_{b-1}\}.$$

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Generalized permutahedra

A *standard permutahedron* is the convex hull in $\mathbb{R}^n$ of $(0, 1, \ldots, n-1)$ and all permutations of its entries. A *generalized permutahedron* $P_n^z(\{z_I\})$, parameterized by collections of real numbers $\{z_I\}$ for $I \subseteq [n]$, is given by

$$P_n^z(\{z_I\}) = \left\{ t \in \mathbb{R}^n : \sum_{i \in I} t_i \geq z_I \text{ for } I \neq [n], \sum_{i=1}^{n} t_i = z_{[n]} \right\}.$$

Theorem (Postnikov '05)

A polytope is a generalized permutahedron if and only if every edge is parallel to a vector $e_i - e_j$.

Theorem (An–T.–Zhang '24)

For $w \in S_n$, $\mathrm{Newton}(D^w)$ is a generalized permutahedron with

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Matroid polytopes

Matroid polytopes, a special type of Newton polytope, were heavily used by Castillo, Cid Ruiz, Mohammadi, and Montaño to prove that double Schubert polynomials have SNP and characterize their Newton polytopes.

Definition

A *matroid* $M = (E, \mathcal{B})$ consists of a finite set E and a nonempty collection of subsets $\mathcal{B}$ of E, which satisfy the *basis exchange axiom*: if $B_1, B_2 \in \mathcal{B}$ and $b_1 \in B_1 \setminus B_2$, then there exists $b_2 \in B_2 \setminus B_1$ such that $B_1 \setminus \{b_1\} \cup \{b_2\} \in \mathcal{B}$.

Definition

The *matroid polytope* $P(M)$ of a matroid $M = ([n], \mathcal{B})$ is

$$P(M) = \mathrm{conv}(\{\zeta^B : B \in \mathcal{B}\}),$$

where $\zeta^B = (\mathbb{1}_{i \in B})_{i=1}^n$ denotes the indicator vector of B.

Matroid polytopes as generalized permutahedra

Proposition (Ardila–Benedetti–Doker '08)

Matroid polytopes $P(M)$ are generalized permutahedra with

$$z_I = r_M([n]) - r_M([n] \setminus I),$$

where $r_M(S) = \max\{\#(S \cap B) : B \in \mathcal{B}\}$.

Definition

For $1 \leq a < b \leq n$, let $M_{ab} = ([n], \mathcal{B})$ be the matroid with $\mathcal{B} = \{\{a\}, \{a+1\}, \ldots, \{b-1\}\}$.

The motivation for defining M_{ab} is the fact that

$$P(M_{ab}) = \mathrm{conv}\{e_a, e_{a+1}, \ldots, e_{b-1}\} = \mathrm{Newton}(x_a + x_{a+1} + \cdots + x_{b-1}).$$

Also by the proposition, we have $P(M_{ab}) = P_n^z(\{\mathbb{1}_{I \supseteq \{a,a+1,\ldots,b-1\}}\})$.

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$$z_I = \sum_{(a,b)\in \mathrm{Inv}(w)} \mathbb{1}_{I \supseteq \{a,a+1\dots,b-1\}}$$

for all $I \subseteq [n]$.

$$\begin{aligned}
\mathrm{Newton}(D^w) &= \sum_{(a,b)\in \mathrm{Inv}(w)} \mathrm{Newton}(x_a + x_{a+1} + \cdots + x_{b-1}) \\
&= \sum_{(a,b)\in \mathrm{Inv}(w)} P(M_{ab}) \\
&= \sum_{(a,b)\in \mathrm{Inv}(w)} P_n^z(\{\mathbb{1}_{I \supseteq \{a,a+1,\dots,b-1\}}\}) \\
&= P_n^z\left(\left\{ \sum_{(a,b)\in \mathrm{Inv}(w)} \mathbb{1}_{I \supseteq \{a,a+1,\dots,b-1\}} \right\}\right).
\end{aligned}$$

Vertices of Newton polytopes

Theorem (An–T.–Zhang '24)

The point $\alpha \in \mathbb{Z}^n_{\geq 0}$ is a vertex of $\mathrm{Newton}(D^w)$ if and only if x^α has a coefficient of 1 in $\mathrm{GW}(w)$.

Theorem (An–T.–Zhang '24)

Given a product q of linear factors in $x_1, x_2, \ldots, x_n$ with all coefficients 1, the point $\alpha \in \mathbb{Z}^n_{\geq 0}$ is a vertex of $\mathrm{Newton}(q)$ if and only if x^α has a coefficient of 1 in q.

Example

$$\begin{aligned} q &= (x_1+x_2)(x_2+x_3)(x_1+x_3)(x_1+x_2+x_3) \\ &= x_1^3x_2 + x_1^3x_3 + 2x_1^2x_2^2 + 4x_1^2x_2x_3 + 2x_1^2x_3^2 \\ &\quad + x_1x_2^3 + 4x_1x_2^2x_3 + 4x_1x_2x_3^2 + x_1x_3^3 + x_2^3x_3 + 2x_2^2x_3^2 + x_2x_3^3 \end{aligned}$$

Vertices: $\{(3,1,0),(3,0,1),(1,3,0),(1,0,3),(0,3,1),(0,1,3)\}$

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Vertices of Newton polytopes (2)

$(1,6)$	$(1,5)$	$(1,4)$	$(1,3)$	$(1,2)$
$(2,6)$	$(2,5)$	$(2,4)$	$(2,3)$	
$(3,6)$	$(3,5)$	$(3,4)$		
$(4,6)$	$(4,5)$			
$(5,6)$				

Step 1: Build a staircase Young diagram with $n = 6$.

Vertices of Newton polytopes (3)

$(1,6)$ 1	$(1,5)$ 0	$(1,4)$ 0	$(1,3)$ 0	$(1,2)$ 0
$(2,6)$ 1	$(2,5)$ 1	$(2,4)$ 0	$(2,3)$ 1	
$(3,6)$ 1	$(3,5)$ 0	$(3,4)$ 0		
$(4,6)$ 1	$(4,5)$ 1			
$(5,6)$ 1				

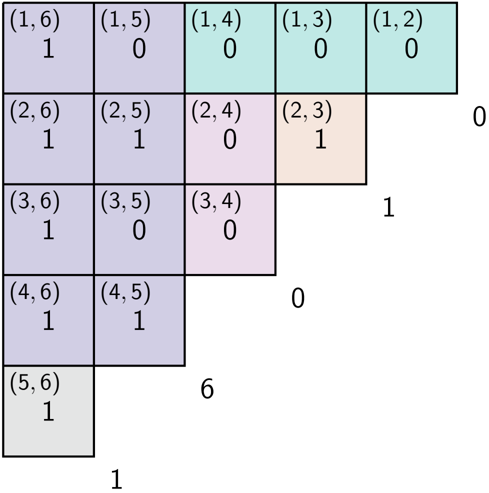
Step 2: When $w = 253641$, the above boxes are filled with 1's.

Vertices of Newton polytopes (4)

$(1,6)$ 1	$(1,5)$ 0	$(1,4)$ 0	$(1,3)$ 0	$(1,2)$ 0
$(2,6)$ 1	$(2,5)$ 1	$(2,4)$ 0	$(2,3)$ 1	
$(3,6)$ 1	$(3,5)$ 0	$(3,4)$ 0		
$(4,6)$ 1	$(4,5)$ 1			
$(5,6)$ 1				

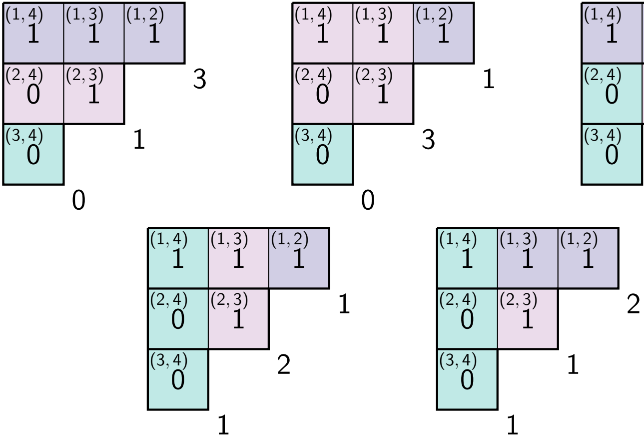
Step 3: We consider a tiling by $n-1$ rectangles.

Vertices of Newton polytopes (5)



Step 4: We find that $\mathrm{Newton}(D^{253641})$ has vertex $(0, 1, 0, 6, 1)$.

Vertices of Newton polytopes (6)



$\mathrm{Newton}(D^{4213})$ has vertices $(3,1,0), (1,3,0), (1,2,1), (2,1,1)$.

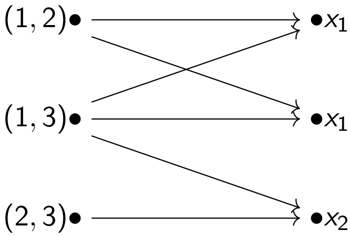
The vanishing problem for dual Schubert polynomials

Theorem (Adve–Robichaux–Yong '21)

For $w \in S_n$, Schubert polynomial $\mathfrak{S}_w$, and $\alpha \in \mathbb{Z}_{\geq 0}^{n-1}$, there is a polynomial-time algorithm to determine whether $\alpha \in \operatorname{supp}(\mathfrak{S}_w)$.

Theorem (An–T.–Zhang '25)

For $w \in S_n$ and $\alpha \in \mathbb{Z}_{\geq 0}^{n-1}$, there is an $O(n^5)$ algorithm to determine whether $\alpha \in \operatorname{supp}(D^w)$.



The network testing the term $x_1^2 x_2$ in D^{321}.

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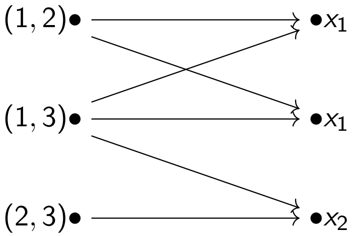

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Further conjectures

Conjecture (An–T.–Zhang '24)

For all Bruhat intervals $[u, w]$ in S_n, D_u^w has SNP, and $\mathrm{Newton}(D_u^w)$ is a generalized permutahedron.

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A chain of implications

$$\text{SNP} \Longleftarrow \text{M-convex} \Longleftarrow \text{Lorentzian}$$

Theorem

An M-convex polynomial has SNP and its Newton polytope is a generalized permutahedron.

Postnikov–Stanley polynomials are Lorentzian

In later work, we resolve our first conjecture.

Theorem (An–T.–Zhang '24)

For all Bruhat intervals $[u, w]$ in Weyl group W, D_u^w is Lorentzian.

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λ-degree

Theorem (Borel–Weil–Bott theorem)

There is an isomorphism of abelian groups between the Picard group $\mathbf{Pic}(G/B)$ *and the weight lattice* Λ *under vector addition.*

Definition

Let D denote the Cartier divisor associated with the line bundle $\mathcal{L}_\lambda$. The λ-*degree* $\deg_\lambda(X)$ of an ℓ-dimensional irreducible subvariety X of G/B is defined as the intersection product $(D^\ell \cdot X)$, which corresponds to the self-intersection number $\int_X (D)^\ell \in \mathbb{Z}$.

For $\lambda \in \Lambda^+$ and the Borel–Weil mapping

$$e : G/B \to \mathbb{P}(V_\lambda),\ gB \to g(v_\lambda),$$

where V_λ is the irreducible representation of the Lie group G with highest weight λ, and v_λ is the highest weight vector, the λ-degree of X represents the intersection number of $e(X)$ with a generic linear subspace of $\mathbb{P}(V_\lambda)$ of codimension $\ell(w)$.

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Proof of Lorentzian-ness

Theorem (An–T.–Zhang '24)

For all Bruhat intervals $[u, w]$ in Weyl group W, D_u^w is Lorentzian.

Proposition

The λ-degree of a (closed) Richardson variety is $(\ell(w) - \ell(u))! D_u^w(\lambda)$.

Example

Let $W = A_2$, $u = 213$, and $w = 321$. Consider the simple roots $\alpha_1 = (1, -1, 0)$ and $\alpha_2 = (0, 1, -1)$, and let $V = \mathrm{span}_{\mathbb{R}}(\alpha_1, \alpha_2)$. We obtain fundamental weights $\omega_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ and $\omega_2 = (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ in V. Let λ be the dominant weight $\omega_1 + \omega_2 = (1, 0, -1)$, so $x_1 = x_2 = 1$. Then

$$\deg_\lambda R_u^w = (\ell(w) - \ell(u))! \cdot D_u^w(1, 1) = 2! \cdot \frac{3}{2} = 3.$$

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- Alex Yong for inviting us to this seminar.

What is a Lorentzian polynomial?

Definition

Let $h(x_1, \ldots, x_n)$ be a degree d homogeneous polynomial, and let $e = d - 2$. We say that h is *strictly Lorentzian* if all the coefficients of h are positive and the quadratic form $\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_e}} h$ has the signature $(+, -, \ldots, -)$ for any $i_1, \ldots i_e \in [n]$.

We say that h is *Lorentzian* if

(1) All the coefficients of h are nonnegative, the support of h is M-convex, and the quadratic form $\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_e}} h$ has at most one positive eigenvalue for any $i_1, \ldots i_e \in [n]$.

What are edge weights in in types other than A?

- W is a Weyl group generated by the simple reflections $s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_r}$ corresponding to the simple roots $\alpha_1, \ldots, \alpha_r$
- $\alpha^\vee$ is the coroot $\frac{2\alpha}{(\alpha,\alpha)}$ corresponding to the positive root $\alpha \in \Phi$
- For any λ in the weight lattice Λ, write $\lambda = x_1\omega_1 + \cdots + x_r\omega_r$, where $\omega_1, \ldots, \omega_r$ are the fundamental weights
- x_i corresponds to the inner product $(\lambda, \alpha_i^\vee)$ induced by the Killing form (i.e. $(\omega_i, \alpha_j^\vee) = \delta_{ij}$)
- For a reflection s_α and a covering relation $u \lessdot us_\alpha$ in the strong Bruhat order of W, the *Chevalley multiplicity* is defined by

$$m(u \lessdot us_\alpha) := (\lambda, \alpha^\vee) = \sum_{i=1}^{r} c_i(\lambda, \alpha_i^\vee) = \sum_{i=1}^{r} c_i x_i,$$

c_i are determined by "how to span a positive root α into the linear combination of simple roots α_i" in terms of coroots

Example: type B_n

- simple roots: $\alpha_i = e_i - e_{i+1}$ for $1 \leq i \leq n-1$; $\alpha_n = e_n$
- positive roots: e_i for $1 \leq i \leq n$ (short roots), $e_i \pm e_j$ for $1 \leq i < j \leq n$ (long roots)
- decomposition of short root $\alpha = e_i$:

$$\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_n$$

- decomposition of long root $\alpha = e_i - e_j$:

$$\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$$

- decomposition of long root $\alpha = e_i + e_j$:

$$\alpha = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} + 2(\alpha_j + \alpha_{j+1} + \cdots + \alpha_n)$$