Log-Concavity in Lie Type A

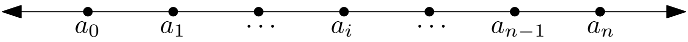
Avery St. Dizier
Michigan State University

UIUC Symposium on (-1)Convexity in Algebraic Combinatorics

Based on joint works with June Huh, Apoorva Khare, Jacob Matherne, and Karola Mészáros.

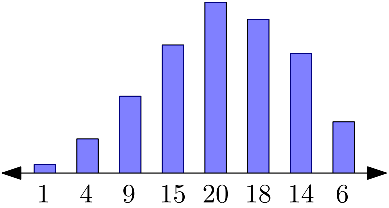
Unimodal Sequences

$\mathbf{a} = (a_0, a_1, \ldots, a_n) \in \mathbb{Z}_{\geqslant 0}^{n+1}$ a combinatorial sequence

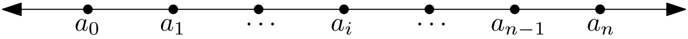


Unimodality:

$$\underbrace{a_0 \leq a_1 \leq \cdots \leq}_{\text{increase}} a_j \underbrace{\geqslant a_{j+1} \geqslant \cdots \geqslant a_n}_{\text{decrease}} \quad \text{for some } j$$



Generalizations



- Log-Concavity:

$$a_i^2 \geqslant a_{i+1}a_{i-1} \text{ for } 1 \leq i \leq n-1$$

- Ultra Log-Concavity:

$$\frac{a_0}{\binom{n}{0}}, \frac{a_1}{\binom{n}{1}}, \ldots, \frac{a_n}{\binom{n}{n}} \qquad \text{is log-concave}$$

$$\text{i.e. } a_i^2 \geqslant \left(1 + \frac{1}{i}\right)\left(1 + \frac{1}{n-i}\right) a_{i+1}a_{i-1}$$

Exercise

For positive sequences:

$$\text{Ultra Log-Concave} \Longrightarrow \text{Log-Concave} \Longrightarrow \text{Unimodal}$$

Some Famous Examples

- Binomial coefficients

$$\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}$$

  are ultra log-concave

- (Butler 1990) q-binomial coefficients $\begin{bmatrix} n \\ 0 \end{bmatrix}_q, \begin{bmatrix} n \\ 1 \end{bmatrix}_q, \ldots, \begin{bmatrix} n \\ n \end{bmatrix}_q$ are log-concave:

$$\begin{bmatrix} n \\ i \end{bmatrix}_q^2 - \begin{bmatrix} n \\ i-1 \end{bmatrix}_q \begin{bmatrix} n \\ i+1 \end{bmatrix}_q \in \mathbb{N}[q].$$

- Strong Mason Conjecture:

$$f_i = \text{number of independent sets of size } i \text{ in a given matroid}$$

  is ultra log-concave (proved by Bränden–Huh and Anari–Liu–Gharan–Vinzant)

Real-Rootedness

Theorem (Newton 1707)

Suppose

$$f(x) := \sum_{j=0}^{n} a_j x^j \in \mathbb{R}[x]$$

is real-rooted. Then $(a_0, a_1, \ldots, a_n)$ *is ultra log-concave.*

Quick proof:

- By Rolle's theorem, $g(x) := \partial^{j-1} f(x)$ is real-rooted.
- So is $h(x) := x^{n-j-1} g(1/x)$.
- So is

$$\frac{2}{n!} \cdot \partial^{n-j-1} h(x) = \frac{a_{j-1}}{\binom{n}{j-1}} x^2 + 2\frac{a_j}{\binom{n}{j}} x + \frac{a_{j+1}}{\binom{n}{j+1}}. \qquad \square$$

Log-Concavity and Products

Theorem (Classical)

If $f = a_0 + \cdots + a_m x^m$ *and* $g = b_0 + \cdots + b_n x^n$ *have positive log-concave coefficients, then so does* fg.

This is not true for unimodal sequences:

$$(1 + x + 2x^2)(1 + x + 3x^2) = 1 + 2x + 6x^2 + 5x^3 + 6x^4.$$

Goal

Describe some log-concavity properties of the maps

- Dimensions

$$\lambda \mapsto \dim V(\lambda)$$

- Characters

$$\lambda \mapsto s_\lambda$$

- Weight multiplicities

$$\alpha \mapsto K_{\lambda,\alpha}$$

- Tensor product multiplicities

$$(\lambda, \mu, \nu) \mapsto c^{\nu}_{\lambda,\mu}$$

- Verma modules

$$\mu \mapsto p(\lambda - \mu)$$

Why?

Why would multiplicities be log-concave ?

Andrei Okounkov

Abstract

It is a basic property of the entropy in statistical physics that is concave as a function of energy. The analog of this in representation theory would be the concavity of the logarithm of the multiplicity of an irreducible representation as a function of its highest weight. We discuss various situations where such concavity can be established or reasonably conjectured and consider some implications of this concavity. These are rather informal notes based on a number of talks I gave on the subject, in particular, at the 1997 International Press lectures at UC Irvine.

My favorite reason

Weight multiplicities are enumerated by integer points of polytopes

For $P_1, \ldots, P_r$ polytopes in $\mathbb{R}^d$ and $x_1, \ldots, x_r \geqslant 0$ the Minkowski sum

$$P = x_1 P_1 + \cdots + x_r P_r$$

has (normalized) volume

$$\sum_{\substack{\alpha \in \mathbb{N}^r \\ |\alpha| = d}} V_{\alpha_1, \ldots, \alpha_r} \frac{x_1^{\alpha_1}}{\alpha_1!} \cdots \frac{x_r^{\alpha_r}}{\alpha_r!}.$$

Theorem (Alexandrov–Fenchel Inequalities)

The coefficients $V_{\alpha_1, \ldots, \alpha_r}$ form log-concave sequences along root directions $\varepsilon_i - \varepsilon_j$.

Goal

Describe some log-concavity properties of the maps

- Dimensions

$$\lambda \mapsto \dim V(\lambda)$$

- Characters

$$\lambda \mapsto s_\lambda$$

- Weight multiplicities

$$\alpha \mapsto K_{\lambda,\alpha}$$

- Tensor product multiplicities

$$(\lambda, \mu, \nu) \mapsto c^{\nu}_{\lambda,\mu}$$

- Verma modules

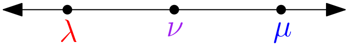
$$\mu \mapsto p(\lambda - \mu)$$

Dimensions of Irreducibles

Lemma (Okounkov 2003)

The function $\lambda \mapsto \dim V(\lambda)$ is log-concave on partitions:

$$(\dim V(\nu))^2 \geqslant \dim V(\lambda)\ \dim V(\mu)$$



By the Weyl dimension formula:

$$\dim V(\lambda) = s_\lambda(1,1,\dots,1) = \prod_{1\leq i<j\leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i}$$

Example:

$$\dim V(2,1,0) = 8 \quad \dim V(4,2,1) = 15 \quad \dim V(6,3,2) = 24$$

Goal

Describe some log-concavity properties of the maps

- Dimensions

$$\lambda \mapsto \dim V(\lambda)$$

- Characters

$$\lambda \mapsto s_\lambda$$

- Weight multiplicities

$$\alpha \mapsto K_{\lambda,\alpha}$$

- Tensor product multiplicities

$$(\lambda, \mu, \nu) \mapsto c^{\nu}_{\lambda,\mu}$$

- Verma modules

$$\mu \mapsto p(\lambda - \mu)$$

Schur Polynomials and Tableaux

Definition

Given a partition $\lambda \in \mathbb{Z}^n_{\geqslant 0}$, the *Schur polynomial* s_λ is defined by

$$s_\lambda(x_1, \dots, x_n) = \mathrm{char}\, V(\lambda) = \sum_{T \in \mathrm{SSYT}(\lambda)} x_1^{(\#1'\mathrm{s\ in\ } T)} \cdots x_n^{(\#n'\mathrm{s\ in\ } T)}$$

$\mathrm{SSYT}(2,1,0)$:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \\ \hline\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \\ \hline\end{array}$$

$$s_{210} = x_1^2 x_2 + x_1 x_2^2 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

Skew Schur polynomial: $s_{\lambda/\mu}$ for $\lambda \supseteq \mu$

Between Schur Polynomials

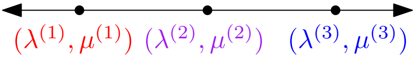
Theorem (Okounkov 1997)

For $i \in \{1,2,3\}$*, suppose* $\lambda^{(i)} \supseteq \mu^{(i)}$ *are partitions with*

$$(\lambda^{(2)}, \mu^{(2)}) = \frac{1}{2}(\lambda^{(1)}, \mu^{(1)}) + \frac{1}{2}(\lambda^{(3)}, \mu^{(3)}).$$

Then

$$s^2_{\lambda^{(2)}/\mu^{(2)}} - s_{\lambda^{(1)}/\mu^{(1)}} s_{\lambda^{(3)}/\mu^{(3)}} \in \mathbb{Z}_{\geqslant 0}[x_1, x_2, \ldots].$$



Moreover, he conjectured *Schur-positivity.*

Between Schur Polynomials

Theorem (Lam–Postnikov–Pylyavskyy 2005)

The following are all Schur-positive

- $\left(s_{\frac{\lambda+\nu}{2}/\frac{\mu+\rho}{2}}\right)^2 - s_{\lambda/\mu}s_{\nu/\rho}$ *(Okounkov)*
- $s_{\mathrm{sort}_1(\lambda,\mu)}s_{\mathrm{sort}_2(\lambda,\mu)} - s_\lambda s_\mu$ *(Fomin–Fulton–Li–Poon)*
- $\prod_{i=1}^{n} s_{\lambda^{[i,n]}} - \prod_{i=1}^{m} s_{\lambda^{[i,m]}}$ *(Lascoux–Leclerc–Thibon)*
- $s_{(\lambda/\mu)\vee(\nu/\rho)}s_{(\lambda/\mu)\wedge(\nu/\rho)} - s_{\lambda/\mu}s_{\nu/\rho}$ *(Lam–Pylyavskyy)*

Goal

Describe some log-concavity properties of the maps

- Dimensions

$$\lambda \mapsto \dim V(\lambda)$$

- Characters

$$\lambda \mapsto s_\lambda$$

- Weight multiplicities

$$\alpha \mapsto K_{\lambda,\alpha}$$

- Tensor product multiplicities

$$(\lambda, \mu, \nu) \mapsto c^{\nu}_{\lambda,\mu}$$

- Verma modules

$$\mu \mapsto p(\lambda - \mu)$$

Kostka Numbers

The *Kostka numbers* are the weight multiplicities

$$\begin{aligned} K_{\lambda\alpha} &= \dim V(\lambda)_\alpha \\ &= \text{coefficient of } x^\alpha \text{ in } s_\lambda \\ &= \#\text{ of SSYT with shape } \lambda \text{ and weight } \alpha \end{aligned}$$

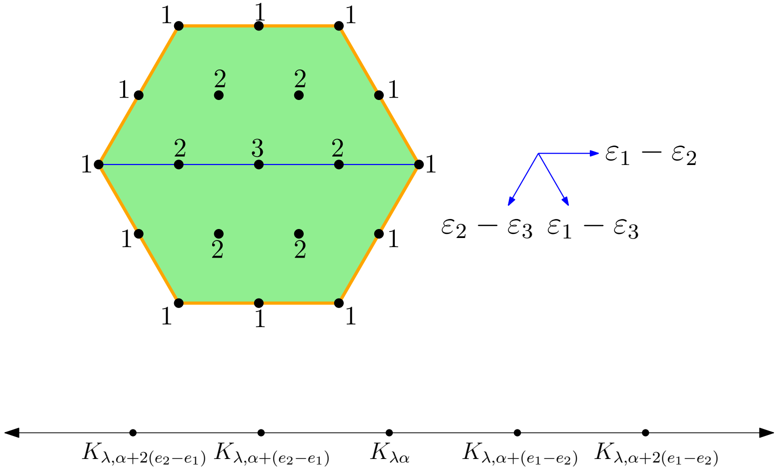
$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$K_{(2,1,0),(1,1,1)} = 2$$

$$s_{210}(x_1, x_2, x_3) = x_1^2x_2 + x_1x_2^2 + 2x_1x_2x_3 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$$

Weight Multiplicity Strings

Weight diagram of $V(4,2,0)$



The SSYT Crystal

Crystal operators on SSYT provides some insight into unimodality of Kostka numbers.

The operator f_i on $\mathrm{SSYT}(\lambda)$ changes an i to an $i+1$ in a tableau T by the recipe:

- map $i \mapsto)$ and $i+1 \mapsto ($
- read parentheses up columns
- iteratively remove matched pairs ()
- change the rightmost) to a (

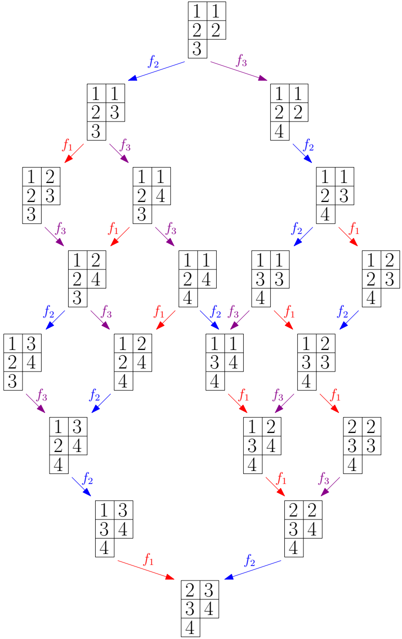
f_2 :

1	2	2	2	3
2	3	4		
3	4	5		
4				

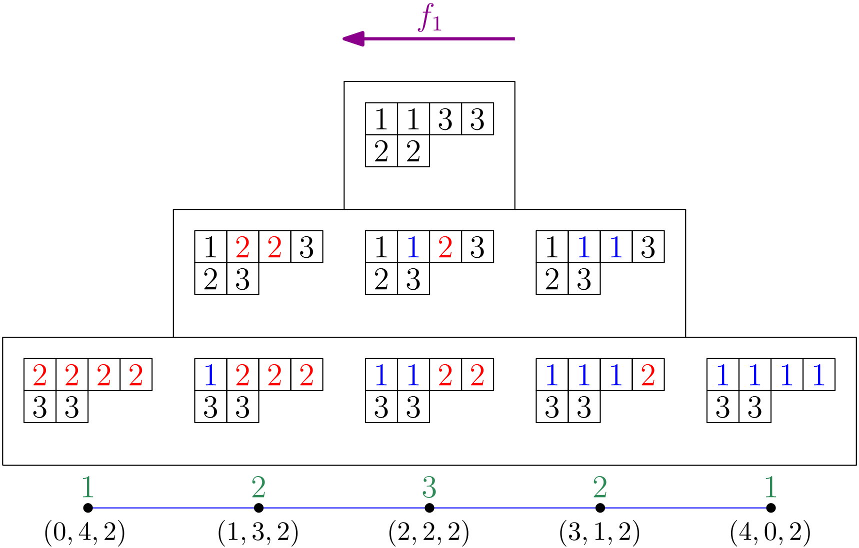
$\longrightarrow \quad ()()))(\to)((\quad \longrightarrow$

1	2	2	3	3
2	3	4		
3	4	5		
4				

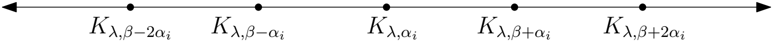
Crystal Operators



Unimodality via Crystals



Log-Concavity of Kostka Numbers?



Proposition

For any partition λ, weight β, and simple root α_i, the sequence $(K_{\lambda,\beta+p\alpha_i})_{p\in\mathbb{Z}}$ is unimodal.

Question

At least partially unimodal, so maybe also log-concave?

$$K_{\lambda\alpha}^2 \geqslant K_{\lambda,\alpha+\varepsilon_i-\varepsilon_j} K_{\lambda,\alpha-\varepsilon_i+\varepsilon_j}$$

Theorem (Huh–Matherne–Mészáros–S. 2019)

Yes! (with Lorentzian polynomials)

Lorentzian Polynomials

Definition (Brändén–Huh 2019)

A homogeneous polynomial f of degree d with nonnegative coefficients is *Lorentzian* if

- $\mathrm{supp}(f)$ equals the set of integer points in a *generalized permutahedron* (M-convexity)
- $\frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_{d-2}}} f$ has at most one positive eigenvalue

What is a polynomial?

Continuous Answer:

A function $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha$$

Continuous log-concavity: $\log(f)$ is a concave function on the domain $\mathbb{R}^n_{>0}$.

If $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_n)$ are log-concave, then so is the product $f(x_1, \ldots, x_n) g(x_1, \ldots, x_n)$

Discrete Answer:

A finite subset S of $\mathbb{N}^n$ together with labels $\alpha \mapsto c_\alpha$

Discrete log-concavity:

$$c_\alpha^2 \geqslant c_{\alpha+\varepsilon_i-\varepsilon_j} c_{\alpha-\varepsilon_i+\varepsilon_j}$$

If $a_0, a_1, \ldots a_n$ and $b_0, b_1, \ldots, b_m$ are positive and log-concave, then so is the convolution $(a_i)_i \otimes (b_i)_i$.

Lorentzian Polynomials

Theorem (Brändén–Huh 2019)

Suppose $f(x) = \sum_{\alpha \in \mathbb{N}^m} c_\alpha x^\alpha$ *is nonzero and* $N(f)$ *is Lorentzian.*

Then:

- $N(f)$ *is log-concave on the positive orthant.*
- $c_\alpha^2 \geqslant c_{\alpha+\varepsilon_i-\varepsilon_j} c_{\alpha-\varepsilon_i+\varepsilon_j}$ *for every* $\alpha \in \mathbb{N}^m$ *and* $i, j \in [m]$.
- *If* $N(g)$ *is also Lorentzian, then so are both* $N(f)N(g)$ *and* $N(fg)$.

More generally: For homogeneous polynomials, Lorentzian is equivalent to being either of

- *strongly log-concave* – f and all derivatives are log-concave
- *completely log-concave* – f and all positive combinations of derivatives are log-concave

Why the quadratic derivatives?

Theorem (Anari–Liu–Oveis–Gharan–Vinzant 2018)

If $f \in \mathbb{R}[x_1, \ldots, x_n]$ is homogeneous of degree d with nonnegative coefficients and

- *for any $\alpha \in \mathbb{Z}^n_{\geqslant 0}$ with $|\alpha| \leq d-2$, $\partial^\alpha f$ is indecomposable;*
- *for any $\alpha \in \mathbb{Z}^n_{\geqslant 0}$ with $|\alpha| = d-2$, the quadratic $\partial^\alpha f$ is log-concave;*

then f is completely log-concave.

Why the quadratic derivaties?

$$c_\alpha{}^2 \geqslant c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j}$$

Proof idea:

$$\frac{\partial^{\alpha-e_1-e_2}}{\partial x^{\alpha-e_1-e_2}} N(f)\Big|_{x_3=\cdots=x_n=0} = \frac{1}{2} c_{\alpha+e_1-e_2} x_1^2 + c_\alpha x_1 x_2 + \frac{1}{2} c_{\alpha-e_1+e_2} x_2^2$$

With at most one positive eigenvalue

$$\det \begin{pmatrix} c_{\alpha+e_1-e_2} & c_\alpha \\ c_\alpha & c_{\alpha-e_1+e_2} \end{pmatrix} \leq 0$$

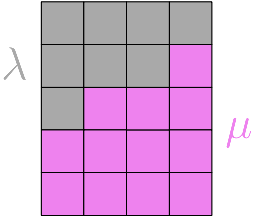
Kostka Numbers

Theorem (Huh–Matherne–Mészáros–S. 2019)

$$N((x_1 \cdots x_n)^{n-1} s_\lambda(x_1^{-1}, \ldots, x_n^{-1})) \text{ *is Lorentzian*}$$

But thanks to symmetry:

$$(x_1 \cdots x_n)^{n-1} s_\lambda(x_1^{-1}, \ldots, x_n^{-1}) \approx s_\mu$$



Goal

Describe some log-concavity properties of the maps

- Dimensions

$$\lambda \mapsto \dim V(\lambda)$$

- Characters

$$\lambda \mapsto s_\lambda$$

- Weight multiplicities

$$\alpha \mapsto K_{\lambda,\alpha}$$

- Tensor product multiplicities

$$(\lambda, \mu, \nu) \mapsto c^{\nu}_{\lambda,\mu}$$

- Verma modules

$$\mu \mapsto p(\lambda - \mu)$$

Littlewood–Richardson Numbers

The *Littlewood–Richardson numbers* are $c^{\nu}_{\lambda\mu} \in \mathbb{Z}_{\geqslant 0}$ defined by

$$s_\lambda s_\mu = \sum_{\nu} c^{\nu}_{\lambda\mu} s_\nu.$$

Classically nonnegative and count various combinatorial objects.

Representation Theory:
$c^{\nu}_{\lambda\mu}$ are the tensor product multiplicities:

$$V(\lambda) \otimes V(\mu) \cong \bigoplus_{\ell(\nu) \leq n} (V(\nu))^{c^{\nu}_{\lambda\mu}}$$

Geometry:
In the cohomology of the Grassmannian $\mathrm{Gr}_{k,n}(\mathbb{C})$:

$$\sigma_\lambda \smile \sigma_\mu = \sum_{\nu \subseteq k \times (n-k)} c^{\nu}_{\lambda\mu} \sigma_\nu$$

Okounkov's Conjecture

Conjecture (Okounkov 2003)

The discrete function

$$(\lambda, \mu, \nu) \mapsto c^{\nu}_{\lambda\mu}$$

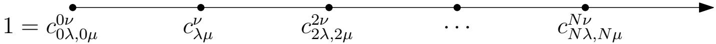
is a log-concave function of λ, μ, ν.

True asymptotically!
(Proved by Okounkov and again by Chindris–Derksen–Weyman)

Implies other true results like...

Saturation

When is $c^{\nu}_{\lambda\mu} > 0$?



If $\log(c^{\nu}_{\lambda\mu})$ is concave, then

$$\{(\lambda, \mu, \nu) \mid c^{\nu}_{\lambda\mu} > 0\} \text{ is convex.}$$

Saturation Conjecture

Theorem (Knutson–Tao 1998)

For any partitions λ, μ, ν,

$$c_{\lambda\mu}^{\nu} > 0 \text{ if and only if } c_{N\lambda,N\mu}^{N\nu} > 0$$

for any $N \geqslant 1$.

The *Horn inequalities* state that $c_{\lambda\mu}^{\nu} > 0$ exactly when (λ, μ, ν) satisfies recursive inequalities of the form

$$\sum_{i \in I} \lambda_i + \sum_{j \in J} \mu_j \geqslant \sum_{k \in K} \nu_k.$$

Okounkov's Conjecture

Theorem (Chindris–Derksen–Weyman 2007)

Okounkov's conjecture is false.

$$c^{0}_{0,0} = 1 \quad c^{\nu}_{\lambda\mu} \quad c^{2\nu}_{2\lambda,2\mu} \quad c^{3\nu}_{3\lambda,3\mu} \quad \cdots \quad c^{N\nu}_{N\lambda,N\mu}$$

Log-concavity would imply

$$(c^{N\nu}_{N\lambda,N\mu})^2 \geqslant c^{(N+1)\nu}_{(N+1)\lambda,(N+1)\mu} c^{(N-1)\nu}_{(N-1)\lambda,(N-1)\mu}.$$

Counterexample:

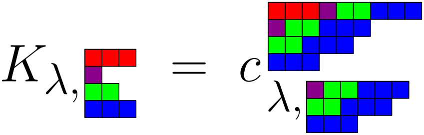
$$\lambda = \mu = 3^{21}2^{21}1^{21}, \quad \nu = 4^{21}3^{42}2^{21}, \quad N = 1$$

Kostka and Littlewood–Richardson

The log-concavity inequalities

$$K_{\lambda\alpha}^2 \geqslant K_{\lambda,\alpha+\varepsilon_i-\varepsilon_j} K_{\lambda,\alpha-\varepsilon_i+\varepsilon_j}$$

are a special case of Okounkov's conjecture!



Theorem (Huh–Matherne–Mészáros–S. 2019)

For partition tuples in the image of the map from Kostka numbers:

$$\left(c_{\lambda,\mu}^{\nu}\right)^2 \geqslant c_{\lambda,\mu+\varpi_{i-1}-\varpi_{j-1}}^{\nu+\varpi_i-\varpi_j} \, c_{\lambda,\mu-\varpi_{i-1}+\varpi_{j-1}}^{\nu-\varpi_i+\varpi_j}.$$

Goal

Describe some log-concavity properties of the maps

- Dimensions

$$\lambda \mapsto \dim V(\lambda)$$

- Characters

$$\lambda \mapsto s_\lambda$$

- Weight multiplicities

$$\alpha \mapsto K_{\lambda,\alpha}$$

- Tensor product multiplicities

$$(\lambda, \mu, \nu) \mapsto c^{\nu}_{\lambda,\mu}$$

- Verma modules

$$\mu \mapsto p(\lambda - \mu)$$

Verma Modules

The *Verma module* $M(\lambda)$ is the universal highest weight module with highest weight λ:

$$M(\lambda) := \frac{U\mathfrak{g}}{U\mathfrak{g}\cdot\mathfrak{n}^+ + \sum_{i\in I}(U\mathfrak{g}\cdot(h_i - \lambda(h_i)))}.$$

By the Poincaré–Birkhoff–Witt theorem:

$$\dim M(\lambda)_\mu = p(\lambda - \mu), \text{ where}$$

$p(\nu)$ is the *Kostant partition function* – the number of ways to write ν as a sum of positive roots.

Example:

$$\begin{aligned} p(2,1,-3) &= |\{2\alpha_1 + 3\alpha_2,\ 2(\alpha_1+\alpha_2)+\alpha_2,\ (\alpha_1+\alpha_2)+\alpha_1+2\alpha_2\}| \\ &= 3. \end{aligned}$$

Verma Modules

Question

Is $\nu \to p(\nu)$ log-concave?

Theorem (Huh–Matherne–Mészáros–S. 2019)

For every $p \in \mathbb{Z}^n$ and $i, j \in [n]$,

$$p(v)^2 \geqslant p(v + \varepsilon_i - \varepsilon_j)p(v - \varepsilon_i + \varepsilon_j).$$

Idea:

- $p(v)$ are the mixed volumes of *flow polytopes*.
- Mixed volumes are log-concave in root directions by the Alexandrox–Fenchel inequalities.

Question

Does this log-concavity hold for a more general family of universal modules over $\mathfrak{sl}_{n+1}(\mathbb{C})$?

Parabolic Verma Modules

For a subset $J \subseteq [n]$, the *parabolic Verma module* is

$$M(\lambda, J) := \frac{M(\lambda)}{\sum_{j \in J} U\mathfrak{g} \cdot f_j^{\lambda(h_j)+1} M(\lambda)_\lambda}.$$

Extremal cases:

- $J = \emptyset$: $M(\lambda, \emptyset) = M(\lambda)$ (the Verma module)
- $J = I$: $M(\lambda, I) = V(\lambda)$ (the finite-dimensional irrep)

Theorem (Khare–Matherne–S. 2025)

The weight multiplicities of any $\mathfrak{sl}_{n+1}(\mathbb{C})$ parabolic Verma module are log-concave along root directions.

Thanks for listening!