Combinatorial Rules for Canonical Decomposition of Quiver Reps

Goal: Define Quivers their representations, canonical decomposition, and present combinatorics for computing the latter in types A_n, D_n

Casey Appleton, June 4th

Quivers and Quiver Representations

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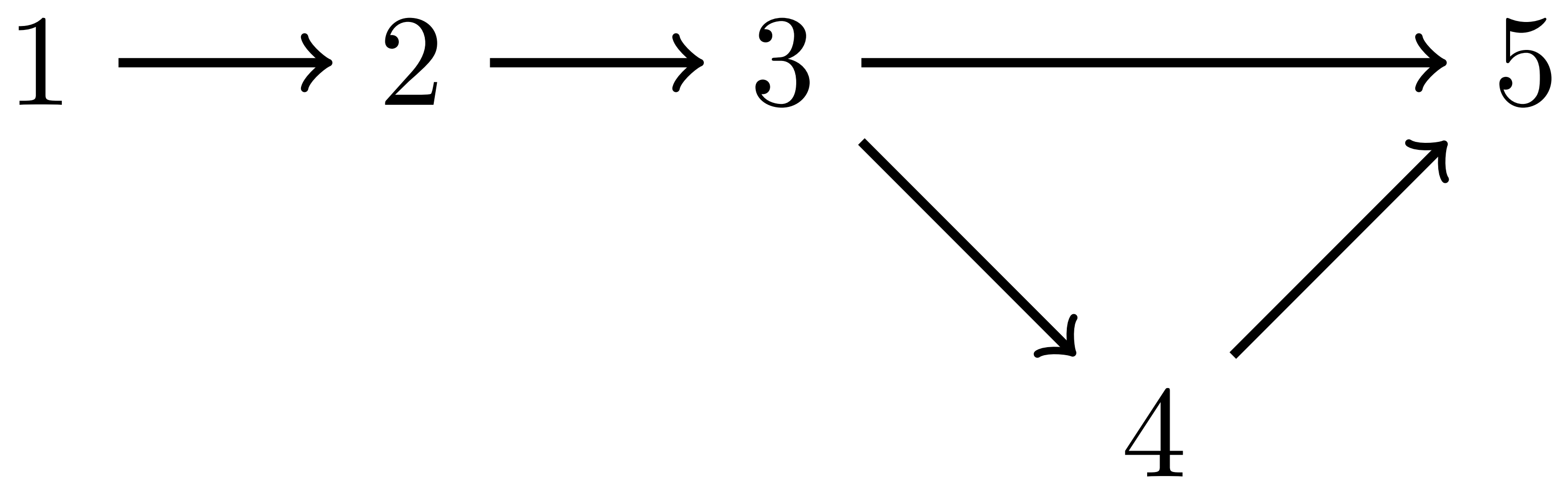

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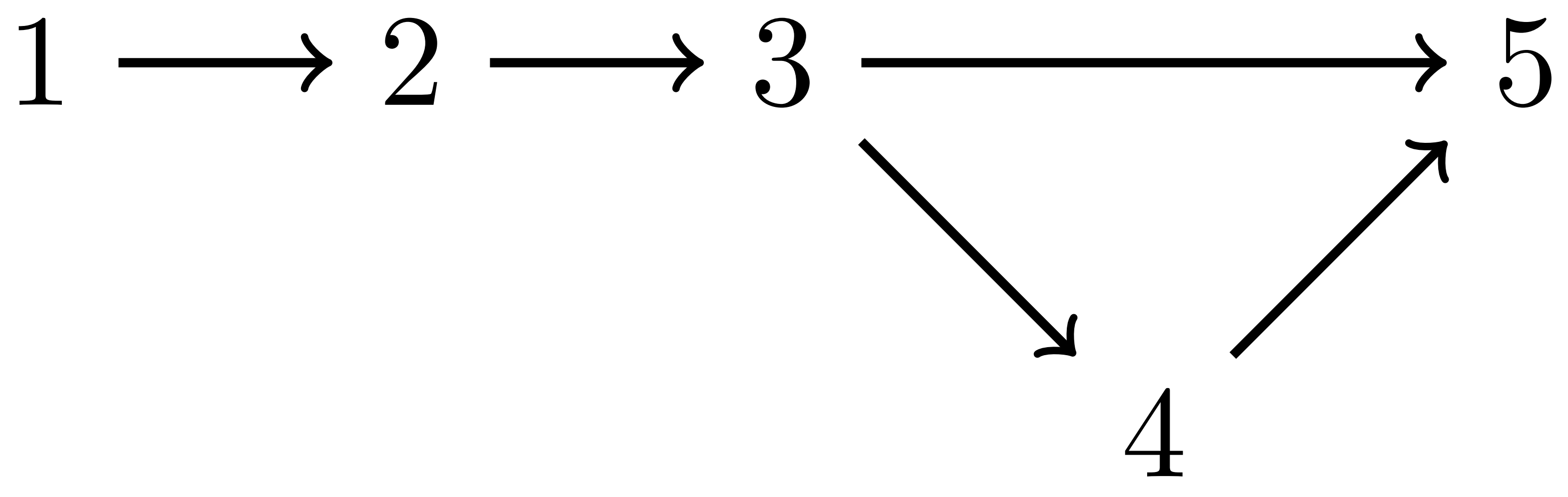

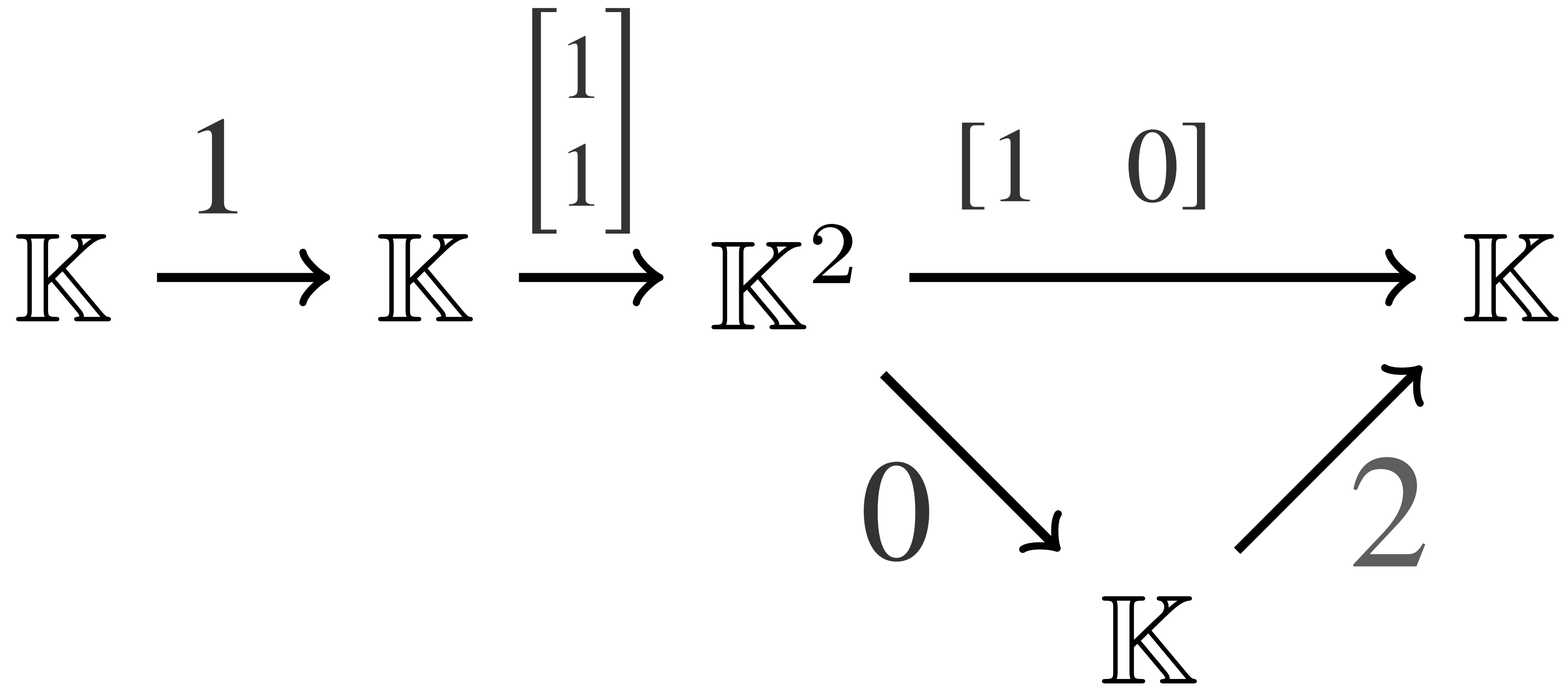

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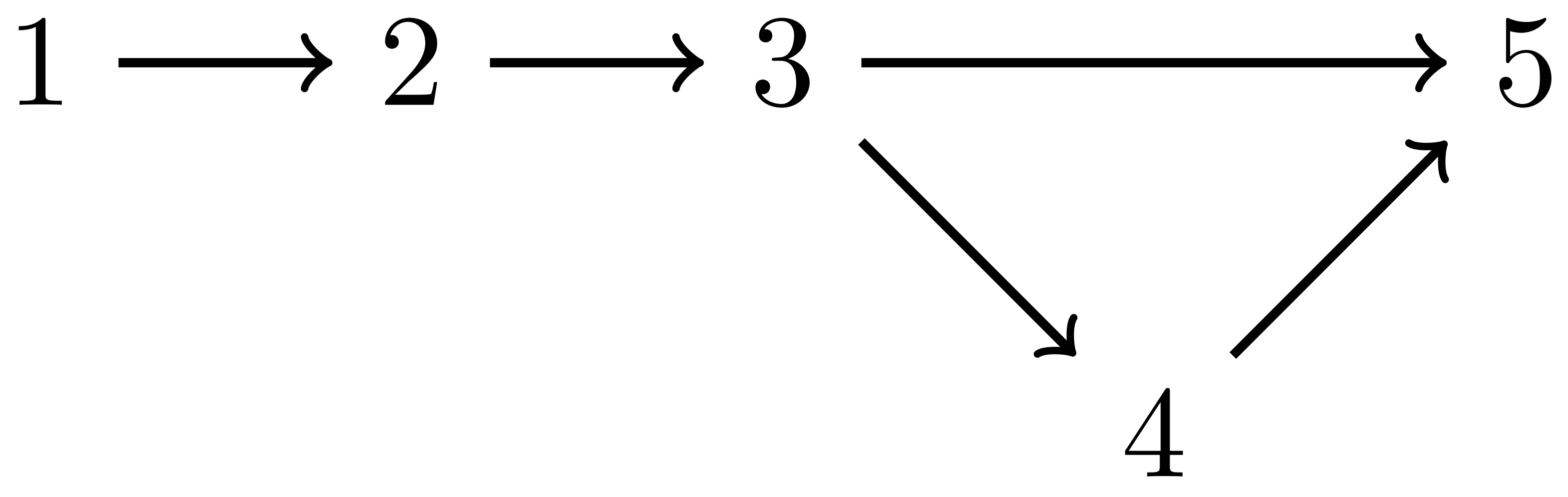
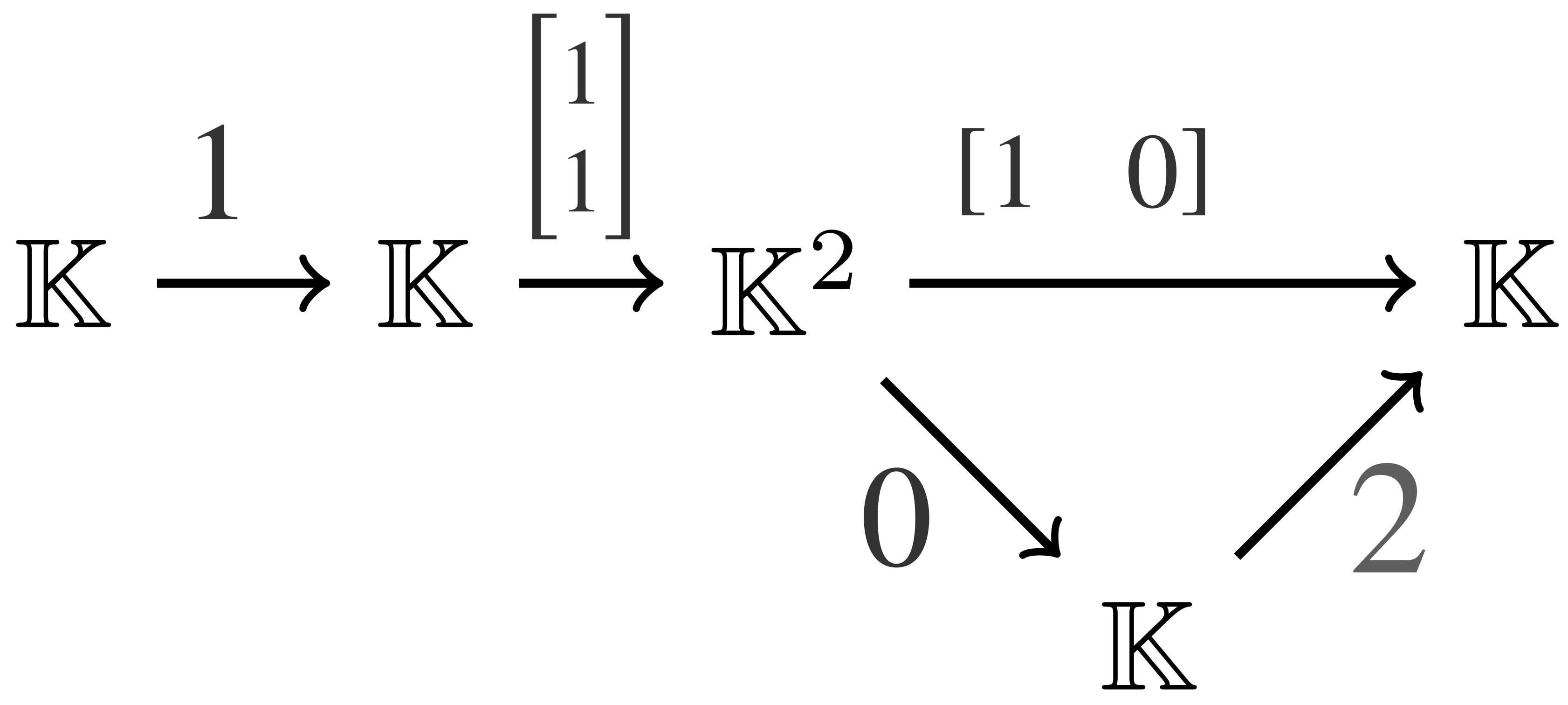
- The dimension vector is the nonnegative integer vector with components $\underline{dim}(V)_i = dim(V_i)$

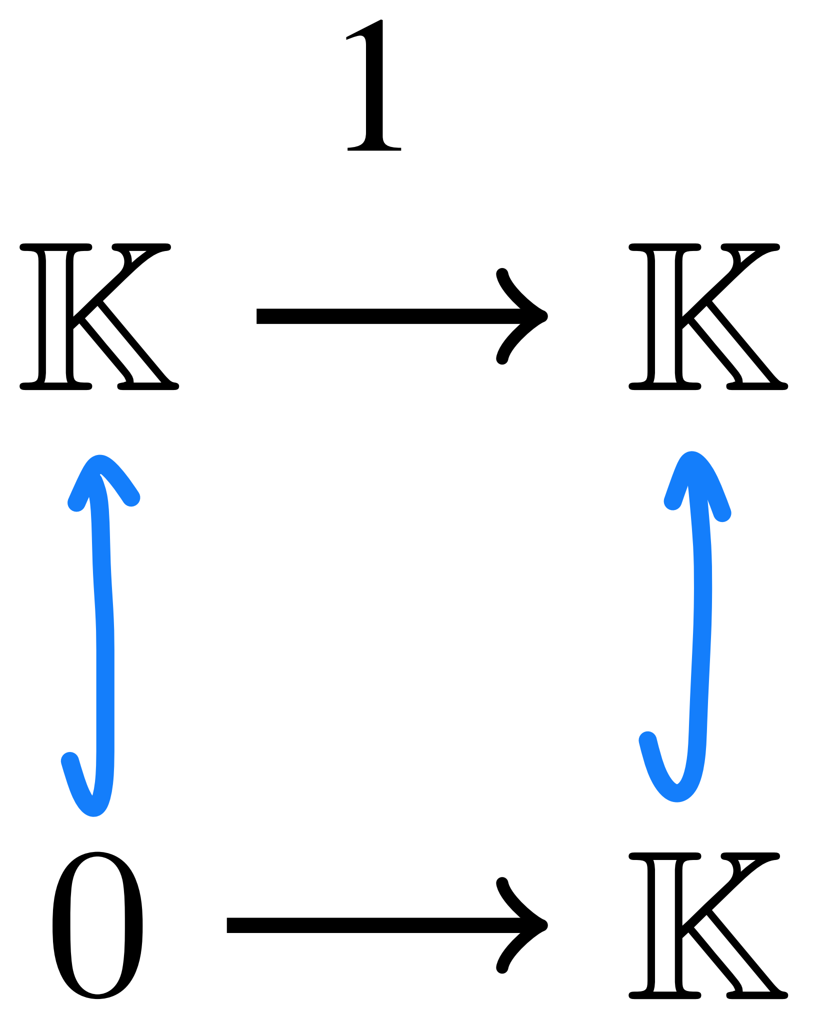




$(1,1,2,1,1)$

Quivers and Quiver Representations
(continued)

- A morphism $\phi : N \to M$ of Q-reps is a Q_0 indexed family of linear maps $\phi_i : N_i \to M_i$ such that squares commute (i.e., $\phi_j \circ M_r = N_r \circ \phi_i$ for $r : i \to j$)



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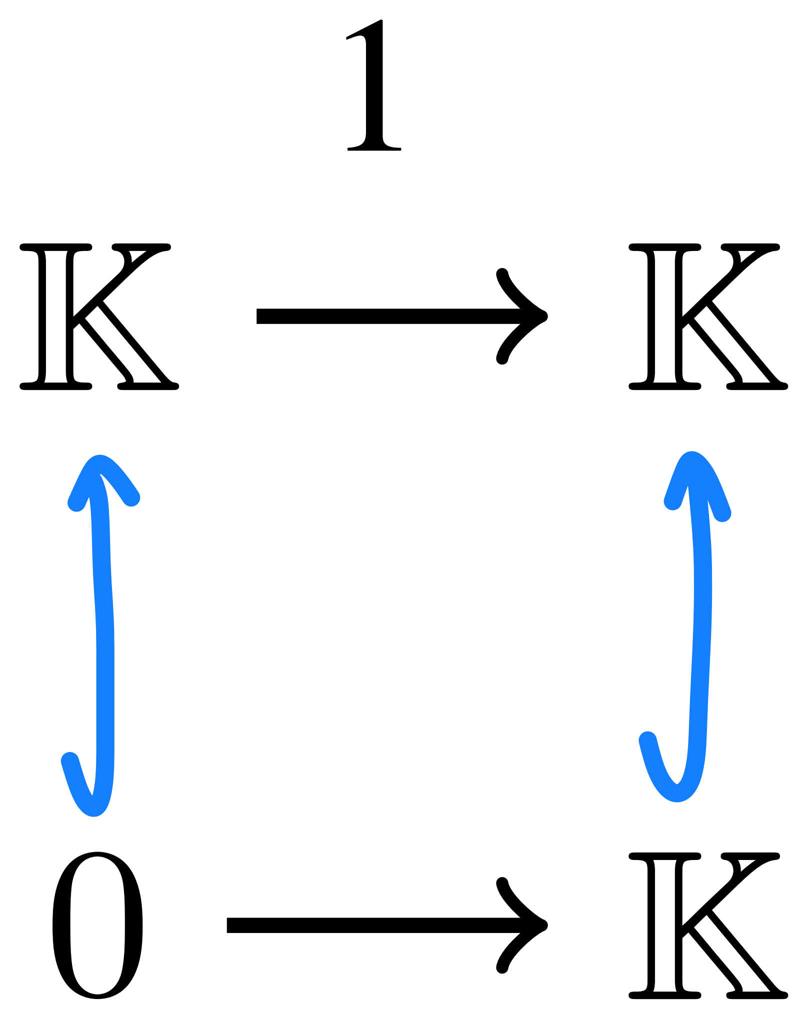

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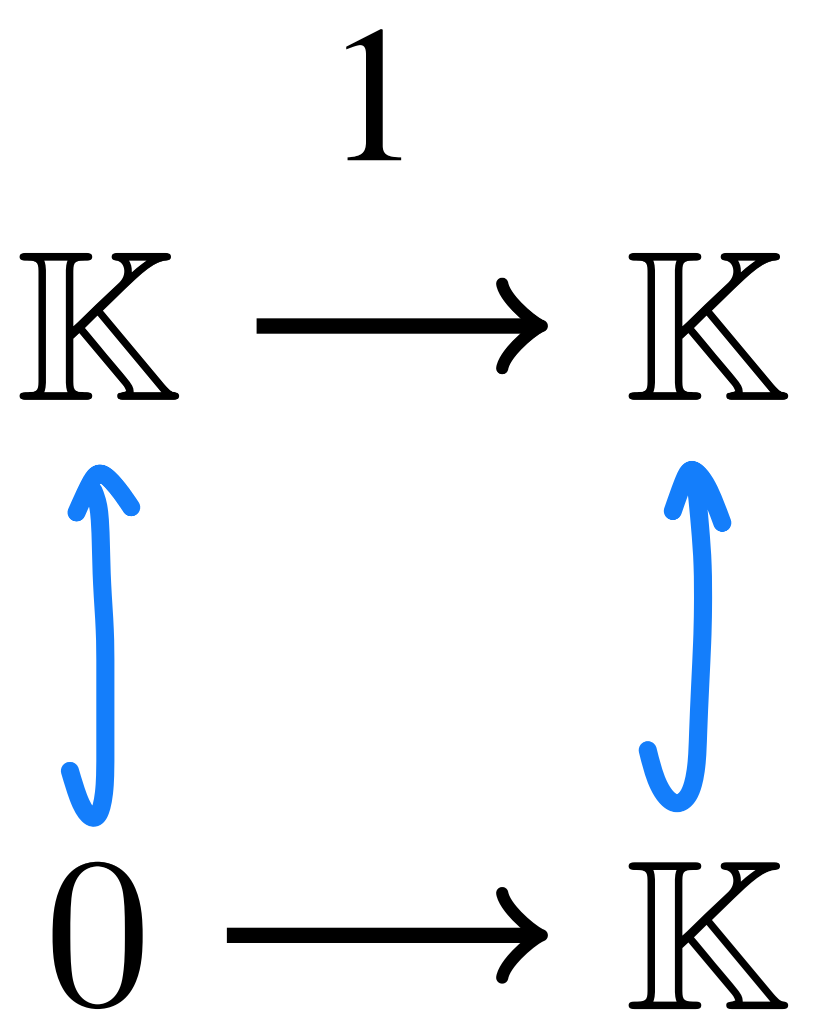

Quivers and Quiver Representations (continued)

- If N and M are representations of Q, their direct sum is given by $(N \oplus M)_i := N_i \oplus M_i$ with $(N \oplus M)_{ij} := N_{ij} + M_{ij} : N_i \oplus M_i \to N_j \oplus M_j$

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- What are A_n, D_n and Dynkin quivers?

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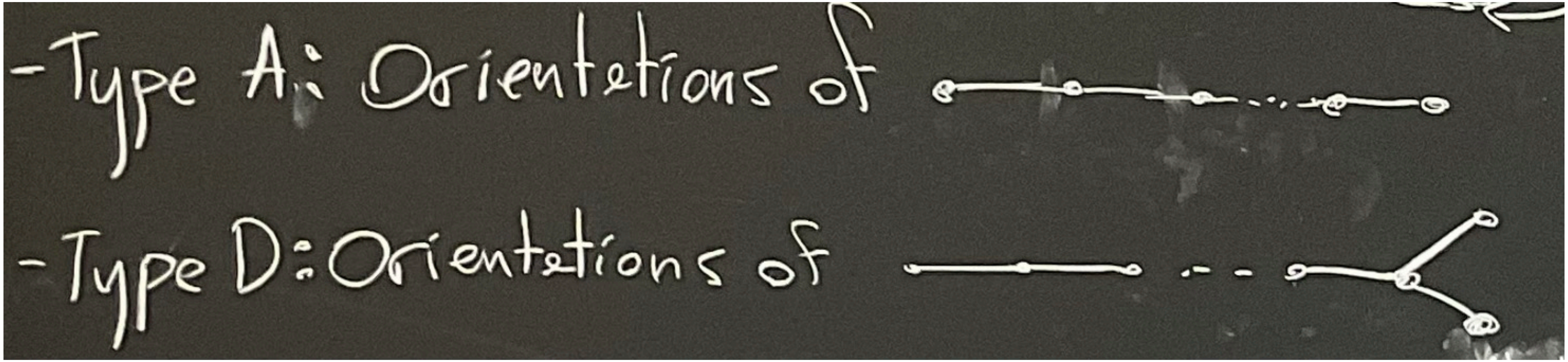
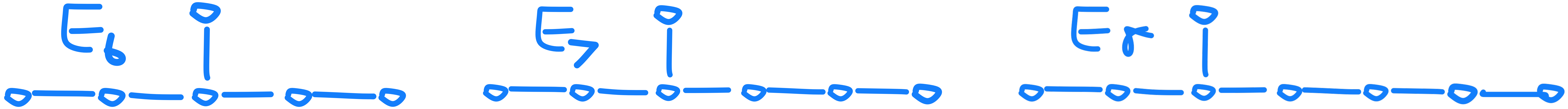
Dynkin Quivers & Diagrams

2 infinite families, A_n & D_n, + 3 exceptions, E_6, E_7, E_8





n denotes the total # of vertices in each case

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- Every representation of Q with dimension vector $\underline{d}$ is isomorphic to a quiver repn constructed in this way

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- Funky Fact: Every indecomposable repn of Q is the canonical decomposition for its own dimension vector.

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- The resulting multiset of pieces/smaller diagrams realizes the canonical decomposition of $(Q, \underline{d})$
- To figure out which pieces correspond to which indecomposables, we can just apply to rule to indecomposable dimension vectors, and see what the resulting piece/diagram looks like

Let's jump into the details of the rules!
(Diagram Construction and dissection)

- Type A_n: https://www.desmos.com/calculator/yn2vabtmxh
- Type D_n: https://www.desmos.com/calculator/uyyq6kztqy