Classical Invariant Theory II

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These scribe notes are based on a talk given by Ian Cavey.

Consider a reductive group acting on a complex vector space, which we can denote as:

$$G \curvearrowright V \simeq \mathbb{C}^n$$

Theorem 0.1 (Hilbert's Finiteness Theorem). *The ring of invariants $\mathbb{C}[V]^G$ is a finitely generated graded algebra.*

There exists finitely many homogeneous invariant polynomials $f_1, ..., f_k$ such that any invariant polynomial can be expressed as a polynomial in $f_1, ..., f_k$:

$$\forall g \in \mathbb{C}[V]^G, g = \sum c f_1^{a_1} ... f_k^{a_k}$$

The ideal I generated by the positive degree invariants in $\mathbb{C}[V]^G$ is called a homogeneous ideal and can be denoted as: $I = \langle \mathbb{C}[V]^G_{>0} \rangle \subseteq \mathbb{C}[V]$.

The action $G \curvearrowright \mathbb{C}[V]$ can be decomposed into $G \curvearrowright \mathbb{C}[V]_d$ where $\mathbb{C}[V]_d$ is the vector space of homogeneous polynomials of degree d. For a group element $g \in G$, the action on a variable x_i is given by a linear combination of the variables x_j, with coefficients a_j depending on g:

$$g : x_i \leftarrow \sum a_j x_j$$

The generators of the ideal I automatically generate the entire ring of invariants $\mathbb{C}[V]^G$.

Theorem 0.2 (Hilbert Basis Theorem). *If I is an ideal in $\mathbb{C}[x_1, ..., x_n]$, then I is finitely generated.*

There exists a finite set of polynomials $g_1, ..., g_k$ such that $I = \langle g_1, ..., g_k \rangle$. A ring is Noetherian if any ascending chain of ideals $\langle g_1 \rangle \subsetneq \langle g_1, g_2 \rangle \subsetneq ... \subsetneq \langle g_1, ..., g_k \rangle$ terminates in a finite number of steps, where $\langle g_1, ..., g_k \rangle = I$.

The vanishing locus $V(I)$ of an ideal I consists of all the points in V where all polynomials in I vanish, that is, they take the value zero. We can denote this as $V(I) \leq V \simeq \mathbb{C}^n$. So for a point $v \in V$, $v \in V(I) \Leftrightarrow \forall f \in I, f(v) = 0$.

As an example, consider $S_n \curvearrowright \mathbb{C}^n$ The invariant ring under this action is generated by the elementary symmetric polynomials $e_1, ...e_n$ where $e_j = \sum\limits_{1 \leq i_1 < ... < i_j \leq n} x_{i_1}...x_{i_j}$. Therefore,

$$\mathbb{C}[x_1, ...x_n]^{S_n} = \mathbb{C}[e_1, ...e_n]$$

Observe that the zero vector $\vec{0} \in V(I)$ always belongs to $V(I)$ for any $G \curvearrowright V$. For the ideals generated by $e_1, ..e_n$, $\{\vec{0}\} = V(I)$, that is, the zero vector is an element of $V(I)$ and $V(I)$ contains only the zero vector. This can be shown by looking a vector $v = (v_1, ..., v_n) \in V(I)$: $e_1(v_1, ..., v_n) = ... = e_n(v_1, ..., v_n) = 0$ which implies that the symmetric polynomials vanish at v. Now, consider the polynomial:

$$(y + v_1)...(y + v_n) = y^n + y^{n-1}(v_1 + v_2 + ... + v_n) + ... + y^{n-j}e_j(v) + ... + e_n(v)$$

If we set the coefficients $e_1(v) = 0, ..., e_n(v) = 0$, the polynomial simplifies to $y^n = 0$, which implies that $v_1 = ... = v_n = 0$. So, v must be $\vec{0}$ and $v(\langle e_1, ..., e_n \rangle) = \vec{0}$ must be the vanishing locus of the ideal generated by the elementary symmetric polynomials $e_1, ..., e_n$.

To understand which points $v \in V$ belong to the vanishing locus $V(I)$, we can rephrase the problem in terms of orbits under the group action. We know that $\vec{0}$ belongs to $V(I)$, since all invariant polynomials in I vanish at $\vec{0}$. The key observation is that if $v \in V(I)$, then the entire orbit $\overline{G \cdot v}$ under the group action is contained in $V(I)$. This is because if a function $f \equiv c$ on a subset $X \subseteq V$ where c is a constant, then $f \equiv c$ on the Zariski closure $\overline{X}$ of that subset.

Specifically, if $\overline{G \cdot v} \ni \vec{0}$ then $G \cdot v \subseteq V(I)$. This is because any positive degree invariant polynomial $f \in \mathbb{C}[V]^G_{>0}$ vanishes at $\vec{0}$ on the orbit $f|_{G \cdot v} \equiv 0$ and also on its closure $\overline{G \cdot v}$.

Theorem 0.3 (Hilbert's Theorem)**.** *The vanishing locus $V(I)$ is the union of all orbits whose closures contain the origin $\vec{0}$:*

$$V(I) = \bigcup_{\overline{G \cdot v} \ni \vec{0}} v$$

The observation described above is only half of the theorem: if the closure of an orbit $\overline{G \cdot v}$ contains $\vec{0}$, then the entire orbit $G \cdot v$ is contained in $V(I)$.

Corollary 0.3.1. *If* $|G| < \infty, V(I) = \vec{0}$

The orbit of any point $v \neq \vec{0}$ under a finite group action cannot have $\vec{0}$ in its closure.

For two points $v, w \in V$ which lie in distinct G-orbits, does there exist an invariant polynomial $f \in \mathbb{C}[V]^G$ such that $f(v) \neq f(w)$? If such a polynomial does not exist, that means that the closures of their orbits intersect, so $G \cdot v, G \cdot w$ are distinct $\Leftrightarrow \overline{G \cdot v} \cap \overline{G \cdot w} = \emptyset$.

Consider the action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^3$ defined by $t \cdot (x, y, z) = (t^{-1}x, y, tz)$. The invariant ring under this action is generated by y and xz, so $\mathbb{C}[x, y, z]^{\mathbb{C}^*} = \mathbb{C}[y, xz]$. For the point $(1, 0, 1)$, it's orbit under the $\mathbb{C}^*$ action traces out a hyperbola in the xz-plane, as shown in Figure 1, because $(t^{-1}x)(tz) = xz$.

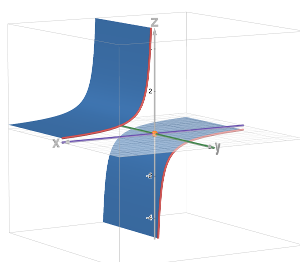

Figure 1: Orbit of $(1, 0, 1)$

The behavior of orbits under the group action $G \curvearrowright V$ and their relation to the origin $\vec{0}$ can be classified into three categories:

1. Unstable points: Points v are considered unstable if the closure of their G-orbit contains the origin $\vec{0}$, i.e.,$\overline{G \cdot v} \ni \vec{0}$.

2. Semistable points: Points v are considered semistable if their G-orbits do not contain $\vec{0}$ in their closure but they can still be distinguished by some invariant polynomials.

3. Stable points: Points v are stable if their stabilizer under the group action is finite and their G-orbit is closed, meaning the point can be distinguished by invariant polynomials. Formally, $v \in V$ is stable if $\text{stab}(v) \leq G$ is finite and $G \cdot v = \overline{G \cdot v}$.

In the example shown by Figure 1, the points can be classified as follows:

- Semistable points: Points that do not lie on the x-axis or z-axis.
- Stable points: Points forming the complement of the yz-plane and xz-plane. These points have both positive and negative weights, with a finite stabilizer.
- Unstable points: Points on the x-axis and z-axis as $V(I) = x\text{-axis} \cup z\text{-axis}$ characterizes the unstable points.

For an infinite group G acting on V, we characterize points in V based off of their stability:

- V^S denotes the set of stable points in V.
- V^{SS} denotes the set of semistable points in V.
- V^u denotes the set of unstable points in V.

Also note the following relations:

- The union of semistable and unstable points covers the entire space V: $V = V^{SS} \sqcup V^u$.
- The set of semistable points contains the set of stable points: $V^{SS} \supseteq V^S$.

For the action of $\mathbb{C}^* \curvearrowright \mathbb{C}^n$ given by $t \cdot (x_1, ..., x_n) = (t^{a_1}x_1, ..., t^{a_n}x_n)$ with $a_1 \leq ... \leq a_n \in \mathbb{Z}$ the space V can be decomposed into three subspaces:

$$V = V_- \oplus V_0 \oplus V_+$$

where:

- V_- corresponds to negative weights $a_1...a_i < 0$, representing components scaled by t^{-1}.
- V_0 corresponds to zero weights $a_{i+1}...a_j = 0$, representing to components invariant under the action.
- V_+ corresponds to positive weights $a_{j+1}...a_n > 0$, representing components scaled by t.

We can classify points under this $\mathbb{C}^*$-action as follows:

- Unstable points: Points in $V_- \cup V_+$, corresponding to components scaled t^{-1} or t (the x-axis and z-axis in the example).
- Semistable points: Points having both positive and negative weights, or any points with 0-weights (not on the x-axis or z-axis in the example).
- Stable points: Points with both positive and negative weights, where the stabilizer is finite. These are points where both the x-coordinate and z-coordinate and non-zero in the example, forming the complement of $V_0 \times (V_- \cup V_+)$

The Hilbert-Mumford criterion simplifies the problem by reducing the problem to considering $\mathbb{C}^* \subseteq G$, a one-parameter subgroup.

Theorem 0.4 (Hilbert-Mumford)**.** *A point, $v \in V$ is unstable (i.e., $\vec{0} \in \overline{G \cdot v}$) $\Leftrightarrow \exists \mathbb{C}^* \subseteq G$ such that $\vec{0} \in \overline{\mathbb{C}^* \cdot v}$*