Classical Invariant Theory

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These scribe notes are based on a talk given by Ian Cavey.

Let

$$G \curvearrowright \mathbb{C}^n$$

This action of G on $\mathbb{C}^n$ induces an action on the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$. Then, for a vector $v \in \mathbb{C}^n$ and a group element $g \in G$, the action of g on v results in a new vector w in $\mathbb{C}^n$. The induced action of G on the coordinate ring $\mathbb{C}[x_1, \dots, x_n]$ is then:

$$(g \cdot \phi)(v) = \phi(g^{-1} \cdot v)$$

where ϕ is a polynomial in $\mathbb{C}[x_1, \dots, x_n]$. For two group elements, $g_1, g_2 \in G$ and a polynomial $\phi \in \mathbb{C}[x_1, \dots, x_n]$:

$$(g_1 \cdot g_2) \cdot \phi(v) = g_1 \cdot (g_2 \cdot \phi)(v) = (g_2 \cdot \phi)(g_1^{-1} \cdot v) = \phi(g_2^{-1} \cdot g_1^{-1} \cdot v)$$

A polynomial $\phi \in \mathbb{C}[x_1, \dots, x_n]$ is G-invariant if $g \cdot \phi = \phi$ for all $g \in G$. The set made from all G-invariant polynomials then forms a subalgebra denoted by:

$$\mathbb{C}[x_1, \dots, x_n]^G$$

where $\phi + \psi, \phi \cdot \psi, c \cdot \psi$ are all G-invariant when ϕ and ψ are invariant polynomials.

For example, consider the symmetric group

$$S_n \curvearrowright \mathbb{C}^n$$

by permuting coordinates. The coordinate ring of symmetric polynomials $\mathbb{C}[x_1, \dots, x_n]^{S_n}$ consists of the polynomials which are invariant under all permutations of $x_1, \dots, x_n$. Symmetric polynomials can be expressed as the elementary symmetric polynomials $e_1(x_1 \dots x_n), \dots, e_n(x_1 \dots x_n)$ where

$e_1(x_1 \ldots x_n)$ is the sum $x_1 + \cdots + x_n$ and $e_n(x_1 \ldots x_n)$ is the product $x_1 \ldots x_n$ and therefore:

$$\mathbb{C}[x_1, \ldots, x_n]^{S_n} = \mathbb{C}[e_1, \ldots, e_n]$$

For some motivation, let's try to understand intrinsic properties, properties that are invariant under the action of a group G. Think of $G \curvearrowright \mathbb{C}^n$ as changes of coordinates. As an example let's use $SL_2 \curvearrowright \mathbb{C}$ which is the projection of changing coordinates. SL_2 consists of 2×2 matrices with determinant 1, and $SL_2 \curvearrowright \mathbb{C}$ by fractional linear transformations:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot x = \frac{\alpha x + \beta}{\gamma x + \delta}$$

We get this fractional linear transformation from $SL_2 \curvearrowright \mathbb{C}$ in the normal way. Given a matrix

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2$$

it acts on a vector $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2$ by usual matrix multiplication:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x & \beta y \\ \gamma x & \delta y \end{pmatrix}$$

When considering $SL_2 \curvearrowright \mathbb{PC}^2 \supseteq \mathbb{C}$ where $y = 1$,

$$\mathbb{PC}^2 = \{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^2 \setminus \{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \} \}$$

In projective spaces, points are equivalent up to scalar multiplication. Therefore:

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \begin{pmatrix} tx \\ ty \end{pmatrix} \text{ for any } t \neq 0$$

When we set $y = 1$ the action of the matrix becomes

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha x & \beta \\ \gamma x & \delta \end{pmatrix} \sim \begin{pmatrix} \frac{\alpha x + \beta}{\gamma x + \delta} \\ 1 \end{pmatrix}$$

So, the group SL_2 acts on the affine line $\mathbb{C}$ by fractional linear transformations.

Considering quadratic forms $ax^2 + bxy + cy^2$:

$$SL_2 \curvearrowright \mathbb{C}_a \oplus \mathbb{C}_b \oplus \mathbb{C}_c$$

where $\mathbb{C}_a \oplus \mathbb{C}_b \oplus \mathbb{C}_c$ are polynomials in x, leads to identifying invariant polynomials. We map the coefficients $(a, b, c) \mapsto ax^2 + bxy + cy^2$. The transformation of the quadratic form under the action of a matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ can be written as:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot (ax^2 + bx + c) = a\bar{x}^2 + b\bar{x} + c$$

where

$$\bar{x} = \frac{\alpha x + \beta}{\gamma x + \delta}$$

Alternatively, we can also write this using coordinates:

$$a\bar{x}^2 + b\bar{x}\bar{y} + c\bar{y}^2$$

where

$$\bar{x} = \alpha x + \beta y,\ \bar{y} = \gamma x + \delta y$$

We deduce this transformation from:

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

Given the quadratic form $ax^2 + bxy + cy^2$, we can represent this in matrix notation as:

$$ax^2 + bxy + cy^2 = (x, y) \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Under the action of SL_2, this quadratic form transforms via the matrix:

$$(x, y) \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

From this, we find that

$$a \mapsto a\alpha^2 + b\alpha\gamma + c\delta^2$$

$$b \mapsto 2a\alpha\beta + b(\alpha\delta + \beta\gamma) + 2c\gamma\delta$$

$$c \mapsto a\beta^2 + b\beta\delta + c\delta^2$$

We also find that the polynomial discriminant $b^2 - 4ac$ remains invariant under the group action:

$$\mathbb{C}[a, b, c]^{SL_2} = \mathbb{C}[b^2 - 4ac]$$