SPECHT MODULES (FROM LINEAR ALGEBRA)

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Let G be a finite group. Define the *group algebra* of G over $\mathbb{C}$ to be the algebra $\mathbb{C}[G]$ with basis $\{g \in G\}$ and multiplication given by

$$(a_1g_1 + \cdots + a_kg_k)(b_1h_1 + \cdots + b_\ell h_\ell) = a_1b_1g_1h_1 + \cdots + a_kb_1g_kh_1 + \cdots + a_kb_\ell g_kh_\ell,$$

where $a_i, b_i \in \mathbb{C}$ and $h_i, g_i \in G$. Let V be a vector space over $\mathbb{C}$, and define

$$\mathrm{Mat}(V) := \{\text{linear maps } V \to V\},\ \mathrm{Mat}(n) := \{n \times n \text{ complex matrices}\}.$$

The following theorem connects the structure of $\mathbb{C}[G]$ with the finite-dimensional representation theory of G.

Theorem 0.1. *Let G be a finite group, and let k denote the number of conjugacy classes of G. Then there exist vector spaces V_i, $1 \leq i \leq k$ so that*

$$\mathbb{C}[G] \simeq_{alg.} Mat(V_1) \times \cdots \times Mat(V_k).$$

This theorem can be proved using the Wedderburn-Artin Theorem and Maschke's Theorem.

Theorem 0.2 (Wedderburn-Artin Theorem). *Let R be a finite-dimensional semisimple algebra over a field k. Then there exist division algebras D_i and integers n_i such that*

$$R \simeq Mat_{n_1}(D_1) \times \cdots \times Mat_{n_k}(D_k).$$

The D_i and n_i are uniquely determined up to isomorphism. If k is algebraically closed, then $D_i = k$ for all i.

Theorem 0.3 (Maschke's Theorem). *Let G be a finite group. Then $\mathbb{C}[G]$ is semisimple. More generally, if k is a field such that the characteristic of k does not divide the order of G, then $k[G]$ is semisimple.*

For example, let $G = S_3$. Then

$$\mathbb{C}[S_3] \simeq_{alg.} \mathrm{Mat}\left(V_{\yng(3)}\right) \times \mathrm{Mat}\left(V_{\yng(1,1,1)}\right) \times \mathrm{Mat}\left(V_{\yng(2,1)}\right),$$

where $V_{\yng(3)}$, $V_{\yng(1,1,1)}$, and $V_{\yng(2,1)}$ are the irreducible representations of S_3, $V_{\yng(3)}$, $V_{\yng(1,1,1)}$ both have dimension 1, and $V_{\yng(2,1)}$ has dimension 2. In terms of matrices, we have that

$$S := \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & e & f \end{bmatrix} : a \neq 0,\ b \neq 0,\ cf - ed \neq 0 \right\} \simeq_{\mathrm{alg}} \mathbb{C}[S_3].$$

The elements of S_3 can be embedded in S in the following manner:

$$() \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (12) \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{S_{-\pi/3}} \end{bmatrix}$$

$$(123) \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{R_{2\pi/3}} \end{bmatrix} \qquad (132) \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{R_{-\pi/3}} \end{bmatrix}$$

$$(23) \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{S_{\pi/3}} \end{bmatrix} \qquad (123) \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & & \\ 0 & 0 & \multicolumn{2}{c}{R_{\pi}} \end{bmatrix}$$

Here,

$$R_\theta := \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

is a rotation matrix and

$$S_\theta := \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

is a reflection matrix. Computing this embedding is difficult in general.

Note that S has a natural action on $\mathbb{C}^4$. Under this action, we can see that

$$V_{\yng(3)} = \left\{ \begin{bmatrix} a \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}, \; V_{\yng(1,1,1)} = \left\{ \begin{bmatrix} 0 \\ b \\ 0 \\ 0 \end{bmatrix} \right\}, \; V_{\yng(2,1)} = \left\{ \begin{bmatrix} 0 \\ 0 \\ c \\ d \end{bmatrix} \right\}.$$

The decomposition in Theorem 0.1 implies that groups with the same finite module categories should have isomorphic group algebras. For instance, while $\mathbb{Z}_2 \times \mathbb{Z}_2 \not\simeq \mathbb{Z}_4$, both $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4$ have 4 irreducible one-dimensional representations (as they are both abelian of order 4). Hence, $\mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2] \simeq \mathbb{C}[\mathbb{Z}_4] \simeq S$, where S is the space of diagonal 4×4 complex matrices. The embedding of $\mathbb{Z}_2 \times \mathbb{Z}_2$ into S is:

$$(1,1) \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (1,1) \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(1,-1) \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad (1,1) \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The embedding of $\mathbb{Z}_4$ into S is:

$$1 \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad -1 \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$i \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \qquad -i \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

The following is an important corollary of Theorem 0.1.

Corollary 0.4. *Let G be a finite group and W a finite-dimensional irreducible representation of G. Then there exists $1 \leq i \leq k$ such that $W \simeq V_i$ as $\mathbb{C}[G]$-modules.*