RE: the exterior algebra and towards the crystal graph

To whom this may concern,

Abigail Price continued her discussion of "highest weights" and the representation theory of $GL(V)$ where here V is a finite-dimensional vector space of dimension n over $\mathbb{C}$. This journal entry reviews what was discussed. I am exposing my knowledge (or lack thereof) here in the hope that the reader will correct me either in person this summer or otherwise.

Let us set some standard notation and terminology. The vector space $\mathrm{Sym}^m(\mathbb{C}^n)$ is the span of the tensors

$$e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_m}, \tag{1}$$

where $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)^t$ is the j-th standard basis vector (with "1" in the j-th coordinate and 0 elsewhere and "t" denotes transpose of a matrix so that e_j is a column vector). In this space we allow commutation, i.e., $e_i \otimes e_j = e_j \otimes e_i$. In this way, there is an isomorphism of vector spaces

$$\mathrm{Sym}^m(\mathbb{C}^n) \cong \{\text{homogeneous polynomials in } x_1, \ldots, x_n \text{ of degree } m\},$$

where the isomorphism sends $e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_m} \mapsto x_{i_1} x_{i_2} \cdots x_{i_m}$.

A more "high-class" way of operating is to work with $\mathrm{Sym}^m(V)$ (i.e., work coordinate and basis-free). One of the advantages of doing so is that one can define $\mathrm{Sym}^m(V)$ in terms of a "universal property" that makes the object unique up to isomorphism. To construct the object, we should really work with the Tensor algebra and take relations so that technically "$e_1 \otimes e_2$" really is an *equivalence class* under the commutation relation, but we're by-passing this discussion here.

An *algebra* is a set S that has the structure of a vector space (vector addition and scalar multiplication) and which is endowed with an additional product. While $\mathrm{Sym}^m(V)$ is a vector space, it is not an algebra. However

$$\mathrm{Sym}^*(V) := \bigoplus_{m \geq 0} \mathrm{Sym}^m(V) \tag{2}$$

is an algebra. Here $\bigoplus$ refers to *direct sum* of vector spaces. It is an infinite dimensional vector space and the product is $\otimes$. Indeed, $\mathrm{Sym}^*(V) \cong \mathbb{C}[x_1, \ldots, x_n]$ as algebras; an element of $\mathrm{Sym}^*(V)$ is a finite linear combination of tensors as in (1) (for various m) – this is all "direct sum" amounts to here.

Another important vector space is $\bigwedge^m(\mathbb{C}^n)$. This is defined as the span of "wedges"

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}. \tag{3}$$

Again, we impose relations where $e_j \wedge e_j = 0$ and $e_i \wedge e_j = -e_j \wedge e_i$. This implies that $\bigwedge^m \mathbb{C}^n$ is the zero vector space if $m > n$. Thus, the *exterior algebra* is the *finite-dimensional* algebra

$$\bigwedge \mathbb{C}^n := \bigoplus_{m=0}^{n} \bigwedge^m(\mathbb{C}^n).$$

Once again, one can go higher class by rather speaking of $\bigwedge V$ and $\bigwedge^m V$.

The algebras $\mathrm{Sym}^*(V)$ and $\bigwedge V$ are the high class way of speaking of "polynomials" and of determinants/rank in linear algebra. As we now discuss, they also provide the main basic examples of irreducible representations of $GL(V)$ = invertible linear operators of V (i.e., isomorphisms of V with itself, a.k.a, $n \times n$ invertible matrices – if one is to be low class).

Recall that a (linear) *representation* of a group G is a group homomorphism

$$\rho : G \to GL(V); \tag{4}$$

it "represents" a group element to an invertible linear transformation $M \in GL(V)$ (a $n \times n$ matrix) that eats vectors from V and spits out vectors of V. That is G "acts" on V by $g \cdot v = \rho(g)v$ (the latter being matrix multiplication). Such actions of G on a vectors space can always be rephrases as a homomorphism (4) and vice versa, so one interchangeably refers to a representation of G as either a homomorphism ρ or simply as V.

The basic question of representation theory is to "determine" the "primes", i.e., the *irreducible representations* of G. These are the representations V for which there is no $\{0\} \subsetneq W \subsetneq V$ that is also acted upon by G. Just to show that this can happen, let $G = \mathfrak{S}_3 = \{123, 213, 132, 231, 312, 321\}$ be the permutations of $\{1, 2, 3\}$. This forms the "symmetric group on three letters". Let G act on $\mathbb{R}^3$ by permuting the coordinates, e.g., $231 \cdot (55, 66, 77) = (66, 77, 55)$. Notice that the set of points $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$ is a vector space, and is fixed under the action of $\mathfrak{S}_3$. Thus $\mathbb{R}^3$ is not an irreducible representation.

The other major problem (although not for this journal entry) is that of "factorization": given a representation of G, tell which what are the irreducibles it contains?

Today, we are mostly interested in the group $GL_n(\mathbb{C})$. In a decent sense, we have "determined" the irreducible representations of $GL_n(\mathbb{C})$ over a century ago. In another sense, it is an area of active research, particularly with respect to "factorization". What Abbie discussed ultimately pertains to both primes and factorization.

Abbie focused on the example of $\bigwedge^2 \mathbb{C}^3$. This has an action of $GL_3(\mathbb{C})$ by

$$g \cdot (v \wedge w) = (g \cdot v) \wedge (g \cdot w).$$

For a sanity check suppose

$$g = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 7 \\ 8 & 6 & 1 \end{bmatrix}, v = (2, 1, 7)^t, w = (9, 1, 1)^t;$$

what is $g \cdot (v \wedge w)$?

To study $\bigwedge^2 \mathbb{C}^3$ or any representation of a "Lie group" such as $GL_3(\mathbb{C})$, it is useful to employ the "most important theorem of mathematics", which determines the representations of $GL_1 = \mathbb{C}^* - \{0\}$, which is called an "algebraic torus"[1]. This most important theorem says that the irreducible representations of $\mathbb{C}^*$ are all of the form $\rho : \mathbb{C}^* \to GL_1$ where $\rho(z) = z^m$ for some integer m. That is, the irreducible representations are all

[1]a torus in math usually refers to a donut, and the complex plane minus the origin is not a donut, but that's the terminology

one-dimensional, and indexed by an integer.[2] The integer m is called the *weight* of the representation. Similarly, if $G = (\mathbb{C}^*)^r$ (Cartesian product) then the (algebraic) irreducible representations all send $(z_1, \ldots, z_r) \in (\mathbb{C}^*)^r \mapsto z_1^{m_1} \cdots z_r^{m_r}$ for some fixed choice of $(m_1, \ldots, m_r) \in \mathbb{Z}^r$.

The standard method to understand a representation W of $GL_n(\mathbb{C})$ is to first use the fact that $n \times n$ invertible matrices include the $n \times n$ diagonal matrices, which are thus isomorphic to the algebraic torus $T = (\mathbb{C}^*)^n$. Thus we can break W into a direct sum of T representations (aka "weight spaces"). Finally, one can hope (somehow) to use this information to "understand" W in complete detail.

For the case of $W = \bigwedge^2 \mathbb{C}^3$, there is a basis given by $e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3$; that is W is three-dimensional. Notice that these three vectors each span weight space. For example, if

$$t = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}, \ t \cdot e_1 \wedge e_2 = t_1 t_2 e_1 \wedge e_2.$$

This means that $e_1 \wedge e_2$ spans the weight space with weight $(1, 1, 0)$. Similarly $e_1 \wedge e_3$ spans the weight space with weight $(1, 0, 1)$ and $e_2 \wedge e_3$ spans the weight space of weight $(0, 1, 1)$. Hence,

$$W \cong V_{(1,1,0)} \oplus V_{(1,0,1)} \oplus V_{(0,1,1)}$$

as T-representations.

To "really" understand W as a G-representation, we would like to understand how the rest of G acts. This is where the discussion got fuzzy for me. What I understood is that since almost every $M \in GL_n$ has an LU-decomposition $M = LU$ where L is lower triangular and U is upper triangular, it suffices to think about the action of upper triangular matrices on W (and thus, by reflection, lower triangular matrices) since we can therefore compose such actions.

At this point, Abbie switched to talking about the Lie *algebra* $\mathfrak{gl}_n$ of $n \times n$ matrices (not necessarily invertible $n \times n$ matrices). She defined the action of $\mathfrak{gl}_n$ on W by

$$M \cdot v := \frac{d}{dt}|_{t=0} e^{Mt} \cdot v,$$

where

$$e^{Mt} := \sum_{k \geq 0} M^k t^k / k!$$

(this "matrix exponential" is a convergent series). Abbie wanted to work with $M \in \{E_{12}, E_{23}\}$ where E_{ij} is the 3×3 matrix with 1 in position (i, j) and 0 elsewhere. She seemed to be getting at W being a $\mathfrak{gl}_3$-representation (a Lie algebra representation) and that E_{12} and E_{23} sent weight spaces to weight spaces such that $\mathrm{span}(e_1 \wedge e_2)$ was either sent to itself or 0.

I guess she is trying to explain how $e_1 \wedge e_2$ is a "highest weight". My understanding is that if W is *any* GL_n irrep, it will have a specific highest weight, and one can inspect

[2]There are numerous caveats to this statement. The first is that the irreducible representations of any abelian group, i.e., a group where the binary operation $\star$ is commutative, i.e, $a \star b = b \star a$, are all one dimensional. The second is that nothing we say actually covers *all* irreducible representations – just the "algebraic" ones that we care about.

W under the action of the Lie algebra to determine which highest weights occur, thus indicating with irreducible representations live in W.

Somehow, the transfer to looking at the Lie algebra $\mathfrak{g}l_n$ versus the Lie group GL_n wasn't explained enough to me. I know that, roughly, knowing representations of one tells you about the other.

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