Generalizing the Three Gap Theorem

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Surrogate Speaker

Student Number Theory Seminar
University of Minnesota, Twin Cities
9 December 2019

The Three Gap Theorem

Circle Interpretation

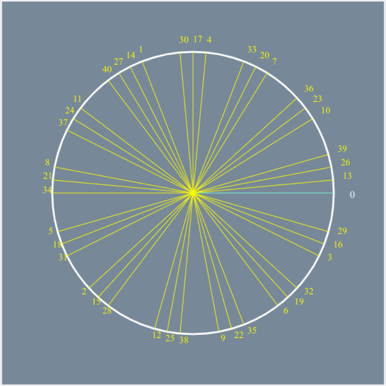


Figure 1: $\alpha \approx 0.31$, $N = 40$

Steinhaus Conjecture (Three Gap Theorem)

Let $\theta \in \mathbb{R}$ and $N \in \mathbb{N}$. If points are marked on a circle at angular displacements of $\theta, 2\theta, \ldots, N\theta$ from a fixed point, then the circumference is partitioned into arcs of at most three distinct lengths.

Proofs given by:

V. T. Sós (1958),
S. Świerczkowski (1958),
and T. van Ravenstein (1988),
among others.

The Three Gap Theorem

Another Perspective: The Fractional Part Function

The fractional part function may be thought of as $f(x) = x \bmod 1$ by identifying 0 and 1 in its codomain. Then for any $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$, there are at most three distinct distances between adjacent pairs of points $f(\alpha), f(2\alpha), \ldots, f(N\alpha)$.

Sawtooth Function Interpretation

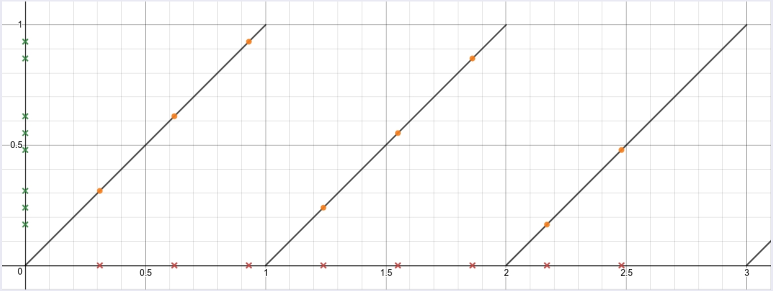


Figure 2: $\alpha = 0.31$, $N = 8$

Motivation

Weyl's Equidistribution Theorem (1909)

For any irrational $\alpha \in \mathbb{R}$, the sequence $\{i\alpha\}_{i \in \mathbb{N}}$ is equidistributed in the interval $[0, 1]$. That is, for any subinterval $(a, b) \subset [0, 1]$,

$$\lim_{n \to \infty} \frac{1}{n} \cdot |\{\{\alpha\}, \{2\alpha\}, \ldots \{n\alpha\}\} \cap (a, b)| = b - a.$$

Illustration

Figure 3: Density of Snow

Existing Generalizations

3d Distance Theorem

Let $\theta, \alpha_1, \ldots, \alpha_d$ be real numbers with $a_1 = 0$ and let $n_1, \ldots, n_d$ be positive integers. The points $n\theta + a_i$ mod 1, where $0 \leq n < n_i$ and $1 \leq i \leq d$, partition the interval $[0, 1]$ into subintervals of at most $3d$ different lengths.

Proofs given by:

F. M. Liang (1979), among others.

Guiding Research Question:

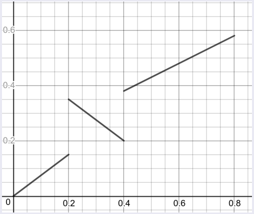
What can we say about the gaps produced by other periodic functions? Does an "n gap theorem" hold for arbitrary periodic functions?

A Particular Result:

Theorem

Let f be a non-negative periodic piecewise linear function with d pieces whose slopes have μ distinct magnitudes and which is injective on its fundamental domain. Then for any $\alpha \in \mathbb{R}$ and $N \in \mathbb{N}$, there are at most $3\mu + d$ distinct gap lengths.

Example



Future of the Project:

Future Directions:

- Finally get this paper over with.
- Never give another presentation on this project again.

Credits

Figure 1:
www.math.brown.edu/~renyi/js/three_gap.html

Figure 3:
www.flickr.com/photos/loren-mooney/26453591605

Trace Functions on Hecke Algebras

December 9, 2019

The Hecke Algebra

Definition

Let (W, S) be a finite Coxeter system, and let q be a parameter. Then a Hecke algebra $\mathcal{H}$ associated to W is the $\mathbb{C}(q)$-algebra with generators $\{T_s | s \in S\}$ and relations

- $T_s^2 = (q-1)T_s + q$
- $T_s T_t T_s \ldots = T_t T_s T_t \ldots$ (m_{st} terms)

Trace Functions

Definition

$\tau : \mathcal{H} \to \mathbb{C}$ is a trace function if $\tau([\mathcal{H}, \mathcal{H}]) = 0$.

It's called a *Markov trace* if it satisfies a further equivariance condition.

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Jones used a Markov trace in his work on knot invariants, which he won a Field's Medal for!

Classification/Weights of Markov Traces

- Type B: Geck-Lambropoulou, Orellana

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- Type B: Geck-Lambropoulou, Orellana
- Type D: Orellana
- Ariki-Koike algebras: Geck-Iancu-Malle

Gomi's Trace

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- Type-independent Markov trace using the Molien series and Lusztig's Fourier transform
- Webster-Williamson gave a geometric interpretation and a uniform proof
- However, this is only a single trace function, and not a classification

Thanks

Thanks for listening!

Representation theory of $GL(2)$ over a finite field

December 9, 2019

Why study this?

- The study of automorphic forms involves representation theory of reductive groups over local fields
- The complex representation theory for $GL_2(\mathbb{F}_q)$ is analagous to the more general theory and is a good entry point

How many irreducible representations are there?

- Number of irreducible representations of $GL_2(\mathbb{F}_q)$ = number of conjugacy classes of $GL_2(\mathbb{F}_q)$
- Analyzing possible eigenvalues of elements of $GL_2(\mathbb{F}_q)$, one finds $q^2 - 1$ conjugacy classes.

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Theorem

- *If $\chi_1 \neq \chi_2$, the resulting representation of $GL_2(\mathbb{F}_q)$ is irreducible.*
- *If $\chi_1 = \chi_2$, the resulting representation is a direct sum of a character and a q-dimensional irreducible representation.*

How many irreducibles do we have so far?

Altogether, from this we get

- $q - 1$ characters
- $q - 1$ irreducible representations of dimension q, called Steinberg representations
- $\frac{1}{2}(q-1)(q-2)$ irreducible representations of dimension $q+1$, called principal series representations

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How many irreducible representations are left? $\frac{1}{2}(q^2 - q)$

The remaining irreducible representations

Theorem

Let E be the quadratic field extension of $\mathbb{F}_q$. The remaining irreducible representations are parameterized by non-decomposable characters of $E^\times$ up to conjugation of the character.

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These representations are the cuspidal representations.

As a check: there are $q^2 - q$ non-decomposable characters of the quadratic extension, so $\frac{1}{2}(q^2 - q)$ of these cuspidal representations. That's how many we had left!

Thanks

Thanks for listening! Happy Holidays!

References

- Complex representations of $GL(2, K)$ for finite fields K, Ilya Piatetski-Shapiro
- Automorphic Forms and Representations, Daniel Bump

Ice Models and Whittaker Functions

A Bridge Between Number Theory and Quantum Groups

December 9, 2019

What's An Ice Model?

Basics

- *A* *state* *of ice is a rectangular grid of tetravalent vertices where each edge is directed*
- *"Ice" rule = each vertex has 2 edges coming in and 2 edges going out*
- *Number rows bottom to top starting at 1, columns right to left starting at 0*

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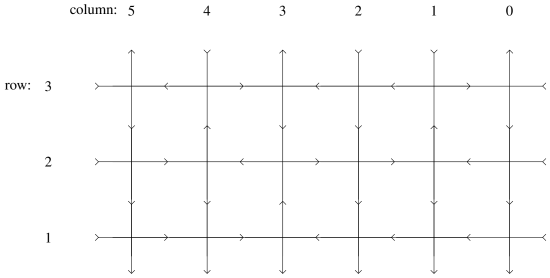

What do I do with an Ice Model?

Turn it into a function!

Definition

We attach a Boltzmann weight $B(v)$ to each vertex v: e.g.

a$_1$	a$_2$	b$_1$	b$_2$	c$_1$	c$_2$
$z_i^{-n_Q\delta(a)}$	1	$g_Q(a)z_i^{-n_Q\delta(a)}$	1	$(1-v)z_i^{-n_Q}$	1

The weight of a state $\mathfrak{s}$ of ice is $B(\mathfrak{s}) = \prod_{v\in\mathfrak{s}} B(v)$.

Given a system of boundary conditions $\mathfrak{S}$, the partition function is

$$Z(\mathfrak{S}) = \sum_{\mathfrak{s}\in\mathfrak{S}} B(\mathfrak{s}).$$

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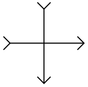	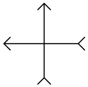	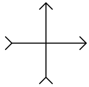	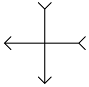	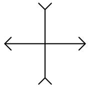	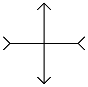
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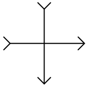	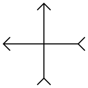	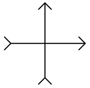	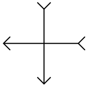	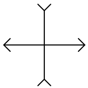	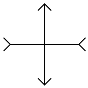
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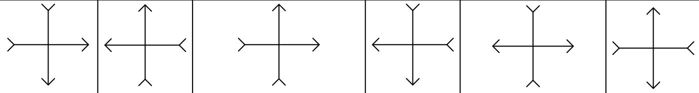

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Why do number theorists care?

Let $G = GL_r(F)$ for a nonarch. local field F.

We can index special elements in T by partitions λ: call each of these t^λ.

We can make a function W^χ called a Whittaker function on any principal series representation of G; it suffices to calculate it on elements t^λ.

Turns out, for a set of boundary conditions $\mathfrak{S}_\lambda$ given by λ and a specific set of weights,

$$W^\chi(t^\lambda) = Z(\mathfrak{S}_\lambda).$$

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Even Better/Future Questions

Even better, this also works for metaplectic extensions of $GL_r(F)$! (using that set of weights earlier, actually...)

Question: does this work for groups of other types?

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Special Values through ages

December 1, 2019

17th Century

Theorem

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

Proof mechanism: relating rational functions to trig functions

$$\arctan(1) = \int_0^1 \frac{1}{1+x^2}dx = \int_0^1 1 - x^2 + x^4 - \cdots$$

Calculus was new!

Bernoulli Challenge

Calculate

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n \geq 1} \frac{1}{n^2}$$

Evaded all mathematicians of the time...
New methods were needed.

18th Century

Theorem

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Proof mechanism: infinite products and analytic (as opposed to geometric) foundations of trig functions.

$$\sum_{n \geq 0} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sin x = x \prod \left(1 - \frac{x^2}{n^2\pi^2}\right)_{n \geq 1}$$

Expand the infinte product and compare the x^3 *terms on both sides.*

Challenges

$$\sum_{n \geq 1} \frac{1}{n^3} = ??$$

No answer yet!

But new discoveries have been made.

19th Century

- Some values of $L(s, \chi)$ using complex contour integration for Dirichlet L-functions

- Eisenstein series and their *rationality* of their Fourier coefficients.

Fourier series was beginning to be understood!

20th Century

- Special values of $\zeta_k(s)$ for number fields $k/\mathbb{Q}$ using rationality of Fourier coefficients of Hilbert-Blumenthal Eisenstein series on GL_2.

- Special values of L-functions attached to holomorphic modular forms using Eisenstein series on LARGER groups. $Sp_{4\times 4}$ is used for GL_2 forms for example.

- Deligne conjectures on why $\zeta(3)$ etc. values could NOT be determined.

21st Century

.... Well .... 80+ years left...

Representations of Renner Monoids

December 9, 2019

The Renner Monoid

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- Bruhat-Renner decomp: $M = \sqcup_{r \in R} BrB$

Example: The Rook Monoid

Definition

The type A Renner monoid, or rook monoid, is the monoid R of 0-1 matrices with an most one entry in each row/column

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Has a nested lattice of diagonal idempotents e:

$$\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ldots & \\ & & & & 0 \end{bmatrix}, \begin{bmatrix} 1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ldots & \\ & & & & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ldots & \\ & & & & 1 \end{bmatrix}$$

Character Tables

Character table of R contains character tables of certain double coset subalgebras eRe. These character tables can be used to compute the whole character table:

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- Munn (Type A)
- Li-Li-Cao (All types)
- H-Marx-Kuo-McDonald-O'Brien-Vetter (Hecke algebras)

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- Is there anything we can glean from the nice geometric structure of monoids?
- Combinatorial interpretations?

Thanks

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