### 0.0.1 Dirichlet L-functions

- Dirichlet (1837) proved there are infinite number of primes in an arithmetic sequence $b, b+m, b+2m, \ldots$ by using Dirichlet L-series $\sum_{n>0} \frac{\chi(n)}{n^s}$, where

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

- **Definition** *Dirichlet character mod m* $\chi : \mathbb{Z} \to \mathbb{C}$ has conditions:

  1. $\chi(n+m) = \chi(n) \quad \forall n \in \mathbb{Z}$
  2. $\chi(km) = \chi(k)\chi(m) \quad \forall k, m \in \mathbb{Z}$
  3. $\chi(n) \neq 0 \Leftrightarrow \gcd(n,m) = 1$
  4. *principal:* $\chi_0(n) = 1 \Leftrightarrow \gcd(n,m) = 1$
  5. *trivial, ie mod 1* $\chi(n) = 1 \; \forall n \in \mathbb{Z}$

  also $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$ extended to $\mathbb{Z}/m\mathbb{Z}$ by $\chi(n) = 0$ for $\gcd(m,n) > 1$

- Has an Euler product

$$\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1}$$

- Tried to follow Legendre, but failed until he started using analytic techniques:

  - Dirichlet made use of

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx$$

    and $s \to 1^+$ in form of a well known identity

$$\int_0^1 x^{k-1} \log^{\rho}\left(\frac{1}{x}\right) dx = \frac{\Gamma(1+\rho)}{k^{1+\rho}}$$

    where $k > 0$ is constant, $\rho > 0$ has $\rho \to 0$.
  - Used complex analysis and the Euler product
  - but did not need analytic continuation.
  - Seems to use $\chi \to$ roots of unity but also needs $\chi(n) = 0$ when $p \mid n$ to eliminate a lot of terms of $\sum \chi(n)/n^s$ to show that

$$\sum \frac{1}{q^{1+\rho}} \to \infty \text{ as } \rho \to 0$$

    where $q = np + m$

- Eisenstein proved analytic continuation and functional equation for a Dirichlet series related to $\zeta$.

- Ernst Kummer (1839,40) introduced $\zeta$ of a cyclotomic field to investigate class number of these fields following Dirichlet
- Riemann (1859) used Poisson summation

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\, dx$$

to show analytic continuation and functional equation of $\zeta$ which is the Dirichlet series with trivial character:

$$\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s) = \xi(1-s)$$

- - Dedekind (1893) extended $\zeta$ to arbitrary number fields of an algebraic extension $K/\mathbb{Q}$ using trivial $\chi$. Dedekind

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p}))^{-1}$$

  $\mathfrak{a}$ non-zero ideal in ring of integers $\mathcal{O}_K$ of $K$ and $\mathfrak{p}$ is prime ideal, $N$ is index $[\mathcal{O}_K : \mathfrak{a}] = |\mathcal{O}_K/\mathfrak{a}|$.
  - Proven by Hecke (1917) to have meromorphic continuation and functional equation.
- **Examples** -The twisted mean square and critical zeros of Dirichlet L-functions
-An explicit lower bound for special values of Dirichlet L-functions
-Several expressions of Dirichlet L-functions at Positive integers
-On asymptotoic properties of the generalized Dirichlet L-functions
-Simultaneous nonvanishing of Dirichlet L-functions and twists of Hecke-Maass L-functions in the critical strip -Explicit bounds on exceptional zeroes of Dirichlet L-functions

-investigation of Dirichlet L-functions of Diaphontine numbers?! (very irrational?!)

### 0.0.2 Hecke L-functions

- A generalization of the Dirichlet L-function and in particular a generalization of Dedekind $\zeta$

$K$ number field,
$v$ non-archimedean place
$\mathcal{O}_K$ ring of integers of $K$,
$\mathfrak{p} \subset \mathcal{O}_K$ prime ideal
$N\mathfrak{p}$ number of elements in finite field $\mathcal{O}_K/\mathfrak{p}$
$|x|_v = |x|_{\mathfrak{p}} = (N\mathfrak{p})^{-ord_{\mathfrak{p}}(x)}$ for $x \in K$

For real embedding $\sigma : K \to \mathbb{R}$ for archimedean $v$ $|x|_v = |\sigma(x)|$.

- Leads to Hecke character (*Grössencharacter*) $\chi_v : K^* \to \mathbb{C}^*$:

$$\chi(x) = \prod_v \chi_v(x)$$

with conditions:

1. $x \in K \subset K_v^*$ implies

$$\chi(x) = 1 \quad \text{product formula}$$

2. all but finite number of $\chi_v$ be *unramified*, ie, trivial on $\{x \in K_v^* \mid |x|_v = 1\}$
3. For unramified place $v$ corresponding to $\mathfrak{p}$, $\chi(\mathfrak{p}) = \chi_v(\varpi_v)$ for uniformizer $\varpi \in K$
4. Ordinary ideal $\mathfrak{a} \subset \mathcal{O}_K$ only included in $\sum$ if product of unramified primes

- Hecke L-function (1916)

$$L_K(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{(N\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left(1 - \chi(\mathfrak{p})(N\mathfrak{p})^{-s}\right)^{-s}$$

where $\mathfrak{a}$, ideals of $\mathcal{O}_K$ are products of prime ideals corresponding to places where $\chi_v$ is unramified.

- $\chi$ trivial, ie., $\chi_v = 1, \forall v$ $L(s, \chi)$ is *Dedekind* $\zeta$ of $K$: $\sum (N\mathfrak{a})^{-s}$. Furthermore, $K = \mathbb{Q}$ becomes Riemann $\zeta$.

- If $\chi$ is finite order $L_K(s, \chi)$ becomes Dirichlet L-function.

- Hecke: express L-function in terms of generalized $\theta$-function, which led to deriving analytic cont., functional equation, boundedness in vertical strips

## 0.1 Modular forms

- Hecke (1936) expanded L-functions into area of Modular forms: theta series:

$$\theta(\tau) = \frac{1}{2} \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau}$$

holomophic in $\mathfrak{H}$, has

$$\theta(\frac{-1}{\tau}) = C(\frac{\tau}{i})^{1/2} \theta(\tau), \quad \theta(\tau + 2) = \theta(\tau)$$

Is a modular form of weight $k = 1/2$ period $\lambda = 2$, $C$ condition for group generated by $\tau \mapsto \tau + 2$ and $\tau \mapsto -\frac{1}{\tau}$, ie has Taylor expansion

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{\frac{2\pi i n \tau}{\lambda}}$$

which implies holomorphic at $\infty$. ($a_0 = 0 \Rightarrow$ cuspform.)

- Hecke: sequence $a_0, a_1, \ldots \subset \mathbb{C}$ $a_n = O(n^d)$, for some $d > 0$. Given $\lambda > 0, k > 0, C = \pm 1$, define:

$$\phi(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

$$\Phi(s) = \left(\frac{2\pi}{\lambda}\right)^{-s} \Gamma(s)\phi(s)$$

$$f(\tau) = \sum_{n=0}^{\infty} a_n e^{\frac{2\pi i n \tau}{\lambda}}$$

  led to

- **Theorem** (Hecke's Converse Thm 1936) Following are equivalent:

  1. $\Phi(s) + \frac{a_0}{s} + \frac{Ca_0}{k-s}$ is an entire function bounded in vertical strips and satisfies functional equation $\Phi(s) = C\Phi(k-s)$
  2. f is a weight $k$ modular form $Mfm(k, \lambda, C)$, period $\lambda$, multiplier condition $C$

- Connects modular forms and L-series/functions, (leads to Wiles discoveries including Fermat's thm)

- Maass forms (1949): non-holomorphic modular forms that are eigenfunctions of Laplacian.

## 0.2 Automorphic forms, Eisenstein Series

- $$E_s(z) = \sum_{\gamma \in (P \cap \Gamma) \backslash \Gamma} \mathrm{Im}(\gamma z)^s$$

  $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}(2) = \Gamma \backslash \mathfrak{H}$, P parabolic, eg., upper triangular. Continues $\zeta$

$$\xi(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s), \quad \xi(s) = \xi(1-s)$$

- Selberg (1962) mero ctn for $E_s$ : $s(1-s)\xi(2s) \cdot E_s$ has analytic ctn to entire fcn of $s$. Fcnl eqn:

$$\xi(2s)E_s = \xi(2-2s)E_{1-s}$$

  Characteristics:

  1. simple pole at $s = 1$ with residue $3/\pi$.
  2. in$0 < \mathrm{Re}(s) < 1/2$ poles at $\rho/2$ where $\rho$ is non-trivial zero of $\zeta(s)$.

- Lots of ways to use Eisenstein series to generate integral representations of L-functions with Euler products, use analytic characteristics of Eisenstein series (analytic continuation, functional equation)

- Colin de Verdière (1982,3) Meromorphic continuation of Eisenstein involves distribution theory including Sobolev spaces, Friedrichs self-adjoint extension of a restriction of a symmetric unbounded operator, eg., the Laplacian

$$\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

- cuspforms are smooth, rapid decay, Eisenstein series is smooth moderate growth.
  - Constant term of Eisenstein series

$$c_P E_s(z) = \int_0^1 E_s(z+t)\, dt$$

$$c_p E(x+iy) = y^s + \frac{\xi(2s-1)}{\xi(2s)} y^{1-s}$$

  - Rankin-Selberg method $f, g$ cuspforms w/ F-series

$$f(z) = \sum_{n>0} a_n e^{2\pi i n z}$$

    then

$$\int_{P\backslash\mathfrak{H}} y^s f(z)\overline{g(z)} y^{2k} \frac{dxdy}{y^2} = (4\pi)^{-(s+2k-1)} \Gamma(s+2k-1) \sum_{n\geq 1} \frac{a_n \overline{b_n}}{n^{s+2k-1}}$$

  - pullbacks of Eisenstein series, eg., Rankin triple product:

$$\mathrm{SL}_2 \times \mathrm{SL}_2 \times \mathrm{SL}_2 \hookrightarrow \mathrm{Sp}_{6\times 6}$$

    holomorphic cuspforms of weight $2k$ for $\mathrm{SL}_2(\mathbb{Z})$: $f, \varphi, \psi$

$$\int\int\int (E\cdot\iota)(z_1,z_2,z_3)\overline{f(z_1)\varphi(z_2)\psi(z_3)}(y_1y_2y_3)^{2k-2}\, dx_1dy_1dx_2dy_2dx_3dy_3$$
$$= \Gamma's \times \zeta's(\text{ constant with}\pi) \times L_{f,\varphi,\psi}(s+4k-1)$$

    has Euler product

- Iwasawa-Tate wraps everything up in the adele's. Garrett MFM notes looks at $\zeta$, Dirichlet L-function in terms of adeles/ideles, eg., $\chi$ is a characger on $\mathbb{J}/k^\times$

### 0.3 Some informal references

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