Topological modular forms

Goals: compute $\pi_k(S^n)$
defined as $[S^k, S^n]$.

Why care? This would ~~to~~ allow you to understand all "nice" spaces.

Chromatic homotopy
built from an analogy

$$\{\text{Cohomology}\} \longleftrightarrow \left\{\begin{matrix}\text{formal} \\ \text{groups}\end{matrix}\right\}$$

TMF: interpolates b/t $\pi_k(S^n)$ and the ring of modular forms.

Generalized cohomology

$\exists$ a functor H^*: Spaces $\longrightarrow$ GrAb
"Singular cohomology"

Thm uniquely classified by the following

1) (Homotopy inv) $f: X \to Y$ is a homotopy equiv of spaces, $H^* f$ is an iso.
2) (Additivity) $H^*(X \vee Y) \cong H^*(X) \oplus H^*(Y)$
3) (Suspension) $H^{*+1}(\Sigma X) \cong H^*(X)$
4) (Exactness) $A \hookrightarrow X$, then $\exists$ LES
5) (Dimension) The point only H^* in degree 0.

Def A generalized cohomology theory is a functor Spaces $\longrightarrow$ grAb satisfying (1)–(4).

Ex (1) complex K-theory.

$K^0(X)$ is the grothendieck group completion of the monoid of complex vector bundles under $\otimes$

$$K^{-n}(X) := K^0(\Sigma^n X)$$

(2) Complex bordism

$$\Omega_n(X) = \left\{ \begin{array}{c} M \\ \downarrow \\ X \end{array} \,\middle|\, \begin{array}{l} M \text{ is a complex} \\ \text{manifold of dim } n \end{array} \right\} \Big/ \sim$$

Group op is $\sqcup$

One of the nice things for H^* is it chern classes.

$\exists$ a map

$$c_1 : Pic(X) \xrightarrow{\sim} H^2(X, \mathbb{Z})$$

<u>Def</u> A cohomology theory E is <u>complex orientable</u> if it has chern classes.

<u>Q</u> How do write $c_1(L \otimes G)$ in terms of $c_1(L)$ and $c_1(G)$.

For H^* : $c_1(L \otimes G) = c_1(L) + c_1(G)$

K^* : $c_1(L \otimes G) = ($ " $) + c_1(L) \cdot c_1(G)$

Ω_* : infinite power series in x, y

$\otimes$ of line bundles is commutative, associative, unital

so any of these power series satifies

① $f(x,0) = x$

② $f(x,y) = f(y,x)$

③ $f(x, f(y,z)) = f(f(x,y), z)$

Def any power series in two variables ~~ax~~ that ①–③ is a formal group law.

FGL

Suppose X is an abelian variety If X was a Lie Group $/\mathbb{R}$, then we could linearize by considering the Lie algebra of X.

you can look at FGL of X, and <u>does</u> recover info about X.

If ~~X~~ G is a Lie group, $\exists$ BCH formula, $X, Y, Z \in \mathrm{Lie}\, G$,

$$e^X e^Y = e^Z$$

$$Z = X + Y + \left\{ \begin{array}{l} \text{higher order things} \\ \text{w/ } [-,-] \end{array} \right\}$$

FGL's have <u>lots</u> of automorphisms, so we get rid of them by considering the underlying formal groups.

$\exists$ a moduli $\mathcal{M}_{FG}$ s.t.

$$\mathrm{Spec}\, R \longrightarrow \mathcal{M}_{FG} \longleftrightarrow \begin{array}{c} \text{FG's} \\ \text{over} \\ R \end{array}$$

$$\begin{tikzcd} \left\{\begin{matrix}\text{Complex}\\ \text{oriented}\\ \text{cohomology}\end{matrix}\right\} \arrow[r, bend left] & \left\{\begin{matrix}\text{formal}\\ \text{groups}\end{matrix}\right\} \arrow[l, bend left, dashed] \end{tikzcd}$$

<u>Thm</u> (LEFT)

If $\hat{\mathbb{G}}$ is a formal group given by $\varphi: \operatorname{Spec} R \longrightarrow \mathcal{M}_{FG}$. If φ is <u>flat</u> then $\exists$ a canonical cohomology theory $E_{\hat{\mathbb{G}}}$.

<u>Elliptic curves</u>

Also organize into a moduli $\mathcal{M}_{ell}$, over $\mathbb{Z}$.

$$\operatorname{Spec} R \longrightarrow \mathcal{M}_{ell}.$$

over $\mathbb{C}$, $\mathcal{M}_{ell} = [\mathfrak{h} // SL_2\mathbb{Z}]$.

Modular forms of weight k are sections of the line bundle $\omega^{\otimes k}$ on $\mathcal{M}_{ell}$.

$$g : \mathcal{M}_{ell} \longrightarrow \mathcal{M}_{FG}$$
$$E/R \longmapsto \hat{E}$$

<u>Fact</u> this is flat. So if you have an elliptic curve E/R,

$$\overbrace{\operatorname{Spec} R \xrightarrow[\text{flat}]{} \mathcal{M}_{ell}} \xrightarrow[\text{flat}]{g} \mathcal{M}_{FG}$$

LEFT, elliptic curve $\rightsquigarrow$ canonical cohomology theory.

~~[illegible]~~ you would like a universal theory. So first step $\mathbb{Z}[\frac{1}{6}]$.

$\exists$ a universal elliptic curve

universal elliptic cohom theory,

Ell^*.

<u>No good!</u> We want 2-, 3-primary info.

$$\begin{tikzcd}
 & \text{Spectra} \arrow[d, "\substack{\text{Goerss}\\ \text{Hopkins}\\ \text{Lurie}}"] \\
\mathcal{M}ell \arrow[ur, dashed, "?"] \arrow[r] & \begin{Bmatrix}\text{complex orientable}\\ \text{cohomology theories}\end{Bmatrix}
\end{tikzcd}$$

bad category

$$\mathrm{TMF} := \Gamma(\mathcal{O}^{top}).$$

one hint for naming,

$$MF_k = \Gamma(w^{\otimes k})$$

<u>connections to number theory,</u>

TMF is special: it's a spectrum.

It has homotopy groups.

$$H^*(\mathcal{M}_{ell}, w^{\otimes *}) \Rightarrow \pi_* \mathrm{TMF}.$$

The bottom row

$$H^0(\mathcal{M}_{ell}, w^{\otimes k}) = MF_k$$

$$\pi_* \mathrm{tmf} \longrightarrow MF_* = \mathbb{Z}[c_4, c_6, \Delta] / (c_4^3 - c_6^2 - 1728\Delta).$$

is an isomorphism, after inverting 2,3.

the kernel is ~~a~~ll torsion info,
the image contains $2c_6$, 24Δ.
but c_6 and Δ not in the image.

There is also a map

$$\pi_* \mathbb{S} \longrightarrow \pi_* \mathrm{tmf}$$

this is an iso on π_0 ~~to~~ through π_6,
"all" interesting torsion ~~is in the~~

of $\pi_* tmf$ is in image of this map,

~~tt~~

tmf knows about 2, 3-primary spheres and ~~all~~ modular forms.

Future directions

$\exists$ filtration $\mathcal{M}_{FG}$ this is the "height" filtration. $h \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$

height	method
0	$H(-, \mathbb{Q})$
1	complex K-theory
2	tmf
3	??? ←
$\vdots$	
∞	$H(-, \mathbb{F}_p)$

One idea: higher dim abelian varieties.

- Tyler Lawson & Mark Behrens:

  TAF $\rightsquigarrow$ n-dim PEL Shimura varieties

all intermediate heights.

- K3 surfaces, height 3. ?

Not well understood