Whittaker functions & Demazure operators

Source: Brubaker, Bump, & Licata

1. Iwahori Whit. fncs
2. Computing ↗ w/operators ⇝ rep'n of Hecke alg
3. Connection to geometry of flag varieties.

1. Iwahori Whit. fncs

Notation: $G = GL_n(\mathbb{Q}_p)$ $\hat{G} = GL_n(\mathbb{C})$

$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$ $B = \begin{pmatrix} & * \\ 0 & \end{pmatrix}$ $N = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$\mathcal{O} = \mathbb{Z}_p$ $\mathfrak{p} = \langle p \rangle$ $q = |\mathcal{O}/\mathfrak{p}|$

$$J = \begin{pmatrix} 0 & 0 \\ 8 & 0 \end{pmatrix}$$

The principal series rep'ns of G are of the form

$$\pi = \mathrm{Ind}_B^G(\tau)$$

char of T

Fix a char. $\psi : N \longrightarrow \mathbb{C}$

~~A~~ Whittaker functional is a linear map

$$\Omega_\tau : \mathrm{Ind}_B^G(\tau) \longrightarrow \mathbb{C}$$

s.t. $\Omega_\tau(\pi(n) f) = \psi(n)\,\Omega_\tau(f)$

Let $M(\tau) := \mathrm{Ind}_B^G(\tau)^J$

The standard basis $\{\Phi_w\}_{w\in W}$ of $M(\tau)$ consists of characteristic functions on J-double cosets & is indexed by $W := S_n$.

We want to calculate:

The Iwahori Whittaker functions:

$$\mathcal{W}_{\tau,\Phi_w}(g) := \Omega_\tau(\pi(g)\Phi_w)$$

2. Demazure-Lusztig operators

A dual connection:

$$z \in \hat{T}(\mathbb{C}) \longleftrightarrow \text{(unram.) chars of } T(\mathbb{Q}_p)$$

$$z = \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \longleftrightarrow \tau_z \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = \prod z_i^{\operatorname{ord}(t_i)}$$

$$\lambda \text{ char of } \hat{T}(\mathbb{C}) \longleftrightarrow a_\lambda \in T(\mathbb{Q}_p)/T(\mathcal{O})$$

$$\lambda \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} = z_1^{\lambda_1} \cdots z_n^{\lambda_n} \longleftrightarrow a_\lambda = \begin{pmatrix} p^{\lambda_1} & & \\ & \ddots & \\ & & p^{\lambda_n} \end{pmatrix}$$

- Suffices to calculate W's on a_λ

- W's will be fncs in $\mathcal{O}(\hat{T})$.

  reg fncs

Def For each $s_i \in W$, the <u>Demazure operator</u> ∂_i on $\mathcal{O}(\hat{T})$ is

$$\partial_i f(z) := \frac{f(z) - \frac{z_{i+1}}{z_i} f(s_i z)}{\left(1 - \frac{z_{i+1}}{z_i}\right)}$$

also consider:

$$D_i = \left(1 - q^{-1} \frac{z_{i+1}}{z_i}\right) \partial_i$$

$$T_i = D_i - 1$$

Thm A To any $w \in W$ & dominant $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$

$$\mathcal{W}_{\mathcal{J}, \Phi_w}(a_{-\lambda}) = (*) \, \mathcal{J}_w (Z^\lambda)$$

Further, $T_i \longmapsto \mathcal{J}_i$ gives a rep'n of the finite Hecke alg $H_{q^{-1}}$ on $\mathcal{O}(\hat{T})$

Remarks The $\mathcal{J}_w$ come from analyzing the effect of intertwining ops $A_w : \mathrm{Ind}_B^G(\tau_z) \longrightarrow \mathrm{Ind}_B^G(\tau_{wz})$ on Ω

· They are closely related to Demazure - Lusztig ops which are derived as endom.'s on equiv. K-theory of flag variety

3. Connection to geometry

· Let $X := \hat{G}/\hat{B}$ $\frac{\text{flag}}{\text{var}}$ (hats implied)

$Y_w = BwB/B$ is a Schubert cell

$X_w = \overline{BwB}/B = \bigcup_{u \leq w} Y_u$

is a Schubert variety

Most X_w's are singular

Let $w = s_{h_1} \cdots s_{h_r}$ be a reduced decomp of w

The Bott-Samelson variety is a certain quotient

$$P_{h_1} \times^B \cdots \times^B P_{h_r} =: Z_w$$

parabolic

e.g. $P_1 = B \cup B s_1 B$

$$= \begin{pmatrix} * & * \\ * & * \\ & & * \end{pmatrix}$$

& $Z_w \longrightarrow X_w$ is a resolution of singularities, w/constant fibers F_u over each Y_u, $u \leq w$.

Thm B

$$\mathcal{D}_w = \sum_{u \leq w} P_{w,u}(q^{-1}) \mathcal{T}_u$$

where $P_{w,u}$ is the Poincaré polynomial of F_u

poly in q^{-1} w/ n^{th} coeff $= H^{2n}(F_u)$

i.e., the relationship between Z_w & Y_u is the same as that between $\mathcal{D}_w$ & $\mathcal{T}_u$. (cool!)

Further remarks:

· This realization of Whittaker fncs also gives efficient proofs of:

  · Casselman - Shalika formula
  · Demazure character formula

& can be used to show that $\mathcal{W}_{\tau_z, \Phi_w}(a \cdot 1)$ is a specialization of non-symmetric Macdonald polys

· (in type A) Hecke alg rep'n $\rightsquigarrow$ R-matrix for a quantum group

a la Andy's talk

$\Big\updownarrow$ a la Ben's class; also see his most recent paper

lattice model