Intro to Klyachko models for GL_n

1. Whittaker models
2. Motivation for study of models
3. Klyachko models

F: finite or p-adic field $\qquad G = GL_n \qquad U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \leq G$

1. Fix a nontrivial character ψ of F. Define a character ψ of U by

$$\psi([u_{ij}]) = \psi(\cancel{a_{1,2} + a_{2,3} + \cdots}\ u_{1,2} + u_{2,3} + \cdots + u_{n-1,n})$$

<u>Def:</u> A Whittaker model of an irreducible rep'n (π, V) of $G(F)$ is the image of an embedding

$$V \longrightarrow \mathrm{Ind}_{U(F)}^{G(F)} \psi.$$

We've encountered Whittaker models for GL_2:

F finite field: $\mathrm{Ind}_{U(F)}^{G(F)} \psi$
- multiplicity free
- all higher dim'l irreps have model

F ~~to~~ non-Archimedean local field: $\mathrm{Ind}_{U(F)}^{G(F)} \psi$
- multiplicity free
- irred. admissible rep'n dim>1 have Whittaker model

Can also define Whittaker models for automorphic rep'ns

↳ leverage decomposition into local rep'ns to deduce global results from local ones.

Uses for Whittaker models

- integral rep'ns of L-functions w/ nice properties
- multiplicity free ⇒ Casselman-Shalika formula
- Fourier expansions of cusp forms

In general, not all irred. admissible higher dim'l rep'ns have Whittaker models.

2. General theme in study of L-functions, write down integral rep'ns of L-functions w/ nice properties to prove analytic continuation and functional equation.

Believed period integrals ~~of~~ related to unique models lead to nice integrals related to L-functions (Piatetski-Shapiro, Furusawa-Shalika)

As said above, not everything has a Whittaker model, but we do still have a multiplicity free property for general n

Klyachko model generalizes the Whittaker model and a symplectic model in hopes of encompassing a wider range of rep'ns.

3. I'll now follow a paper of Offen & Sayag: "Global mixed periods and local Klyachko models for the general linear group".

Write $n = r + 2k$.

Let

$$H_{r,2k} = \left\{ \begin{pmatrix} u & * \\ 0 & h \end{pmatrix} \in G_n : u \in U_r, h \in Sp_{2k} \right\}$$

where $Sp_{2k} = \left\{ g \in G_{2k} : {}^t g \begin{pmatrix} & w_k \\ -w_k & \end{pmatrix} g = \begin{pmatrix} & w_k \\ -w_k & \end{pmatrix} \right\}$. $w_k = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$

Extend ψ to a character of $H_{r,2k}$ by $\psi\begin{pmatrix} u & * \\ 0 & h \end{pmatrix} = \psi(u)$.

<u>Def</u>: The Klyachko model $M_{r,2k}$ is

$$M_{r,2k} = \mathrm{Ind}_{H_{r,2k}(F)}^{G(F)} \psi.$$

$M_{n,0}$ Whittaker model

$M_{0,n}$ (n even) symplectic model

$M_{r,2k}$ is a mixed Whittaker-symplectic model

An irred. admissible rep'n admits the model $M_{r,2k}$ if it can be embedded in this space.

- <u>Klyachko models were first introduced by Klyachko in '84 over finite field</u>: F finite field

$$M = \bigoplus_{k \geq 0}^{[\frac{n}{2}]} M_{n-2k,2k}$$

is a direct sum of all irreps each w/ mult. one.

(Inglis-Saxl '91)

$\to$ M is a <u>Gelfand model</u>:

1. existence
2. disjointness
3. uniqueness

w/ success over finite fields, may want to investigate over p-adic.

- Heumos & Rallis '90 were first to study ~~of~~ Klyachko models over p-adic field: F p-adic

disjointness } Offen-Sayag '08
uniqueness }
existence? ✓

Even in GL_3 exist irred. admissible rep'ns that admit no Klyachko model.

But every ~~smooth~~ admissible (Thanks, Andy!) irred. unitary rep'n admits a Klyachko model.

Tadic's classification of unitary rep'ns:

δ: irred smooth square integrable rep'n of G_r

$t \in \mathbb{Z}^+$

$\delta[\frac{1-t}{2}] \times \delta[\frac{3-t}{2}] \times \dots \times \delta[\frac{t-1}{2}]$ has a unique irred. subrep'n $U(\delta,t)$.

Notation: $\rho[\alpha] = |\det|^{\alpha}\rho$

$\sigma_1 \times \sigma_2$ rep'n of $G_{r_1+r_2}$ parabolically induced from $\sigma_1 \otimes \sigma_2$.

B = collection of $U(\delta,t)$ and $U(\delta,t)[\alpha] \times U(\delta,t)[-\alpha]$ $0<\alpha<\frac{1}{2}$

<u>Theorem</u> (Tadic '86): The unitary rep'ns are exactly of the form

$$\sigma_1 \times \cdots \times \sigma_t$$

w/ $\sigma_1, \ldots, \sigma_t \in B$.

Offen & Sayag use results on the purely symplectic model and highest derivatives of rep'ns to show the following

<u>Theorem</u> (Offen-Sayag '07)

$$U(\delta_1', 2m_1')[\alpha_1'] \times \cdots \times U(\delta_{q'}', 2m_{q'}')[\alpha_{q'}']$$
$$\times\, U(\delta_1, 2m_1+1)[\alpha_1] \times \cdots \times U(\delta_q, 2m_q+1)[\alpha_q]$$

admits the model $M_{r,2k}$ w/

$$r = r_1 + \cdots + r_q$$
$$k = m_1 r_1 + \cdots + m_q r_q + m_1' r_1' + \cdots + m_{q'}' r_{q'}'$$

- Klyachko models also studied over $\mathbb{R}$ & $\mathbb{C}$.
- conjectured by Heumos '93 that irreducible unitary automorphic rep'n of $GL_n(\mathbb{A}_F)$, F a number field has a unique model.

Sources:

- "Models and periods for automorphic forms on GL_n" Heumos
- "Global mixed periods and local Klyachko models for the general linear group" Offen & Sayag
- "A tour of p-adic representation theory of GL_2" Katy Weber ✓ Notes for Katy's talk last semester on SNT site
- "Automorphic forms and representations" Bump.