Intro to the geometric Satake equivalence

Last time:

Classical Satake:

$$\mathcal{H}(G,K)\otimes\mathbb{Z}[q^{\pm\frac{1}{2}}] \xrightarrow{\sim} K(\operatorname{Rep}\check{G})\otimes\mathbb{Z}[q^{\pm\frac{1}{2}}]$$

What this accomplished:

- Some idea of why $\check{G}$ appears:
  - Something something cocharacters!
  - $\coprod_{\lambda\in X_*(T)^+} K\lambda(t)K$ plays some role.
- Roadmap for future studies.

What this doesn't accomplis[h]

- Give a classification-free relationship $G \leftrightarrow \check{G}$

  e.g.

$$\begin{tikzcd} & X_*(T)^+ \arrow[dl] \arrow[dr] & \\ H(G,K)\otimes\mathbb{Z}[q^{\pm\frac{1}{2}}] & & K(\operatorname{Rep}\check{G})\otimes\mathbb{Z}[q^{\pm\frac{1}{2}}] \end{tikzcd}$$

- Construct $\check{G}$-rep'ns.

  i.e. info is lost along

$$\operatorname{Rep}\check{G} \xrightarrow{\text{"forgetful"}} K(\operatorname{Rep}\check{G})\otimes\mathbb{Z}[q^{\pm\frac{1}{2}}].$$

What we really want:

(unrealistic)

$$\begin{tikzcd} & \operatorname{Rep}\check{G} \arrow[d] \\ H(G,K)\otimes\mathbb{Z}[q^{\pm\frac{1}{2}}] \arrow[ur, dashed, "\exists?"] \arrow[r] & K(\operatorname{Rep}\check{G})\otimes\mathbb{Z}[q^{\pm\frac{1}{2}}] \end{tikzcd}$$

More realistic idea: construct "spherical Hecke category" $\mathcal{H}$ such that $\mathcal{K}(\mathcal{H}) \cong \mathcal{H}(G,K) \otimes \mathbb{Z}[q^{\pm\frac{1}{2}}]$ and

i.e.

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\text{(equivalence)}"] \arrow[d, "K(-)\otimes \mathbb{Z}[q^{\pm\frac{1}{2}}]"'] & \operatorname{Rep} \check{G} \arrow[d, "K(\,)\otimes \mathbb{Z}[q^{\pm\frac{1}{2}}]"] \\ \mathcal{H}(G,K) \otimes \mathbb{Z}[q^{\pm\frac{1}{2}}] \arrow[r, "\sim"] & \mathcal{K}(\operatorname{Rep} \check{G}) \otimes \mathbb{Z}[q^{\pm\frac{1}{2}}] \end{tikzcd}$$

Desired properties of $\mathcal{H}$

- Constructed from G!
- $\oplus, \otimes$ compatible with that of $\operatorname{Rep} \check{G}$ that also induce $+, *$ on $\mathcal{H}(G,K) \otimes \mathbb{Z}[q^{\pm\frac{1}{2}}]$
  - $\otimes$ symmetric!
- $\mathcal{H} \longrightarrow \operatorname{Rep} \check{G}$ "natural"

Construction of $\mathcal{H}$: milk definition of $\mathcal{H}(G,K)$ for all it's worth!

idea: $\mathcal{H}(G,K) = \operatorname{Fun}({}_{G(\mathcal{O})}G(F)_{G(\mathcal{O})}, \mathbb{Z})$,
replace functions with some (generalization of) a sheaf

- Grothendieck's fonctions-faisceaux dictionary.

<u>Idea</u>: Construct "$G(F)/G(\mathcal{O})$" and construct sheaves there with an equivariance condition to get at $G(\mathcal{O})\backslash G(F)/G(\mathcal{O})$

The affine Grassmannian

- We do the function version—p-adic is similar but needs more adjectives and Witt vectors.

- Let k be a field (some functorial goon for a moment)
- Define $Gr_G : k\text{-alg} \to Sets$ in two steps
- Let $\mathcal{L}^+G : k\text{-alg} \longrightarrow Sets$

$$A \longmapsto G(A[\![t]\!])$$ (like a loop group)

and $\mathcal{L}G : k\text{-alg} \longrightarrow Sets$

$$A \longmapsto G(A((t)))$$

Then $Gr_G = [\mathcal{L}G / \mathcal{L}^+G]$ in the fpqc topology.

Unwinding what's going on:

- Consider k-points

$$Gr_G(k) = G(k((t)))/G(k[\![t]\!])$$
$$= G(F)/G(\mathcal{O})$$

- Coset space from before
- Example: $Gr_{GL_n}(k) = \{ k[\![t]\!]^n\text{-lattices in } k((t))^n \}$

Geometry

· ind-scheme structure

$Gr_G = \varinjlim X_i$, X_i projective ordinary schemes / k.

i.e. Gr_G is some sort of "infinite dimensional" union of projective schemes/k

Similar idea to some $\mathbb{P}^\infty_k$ (cf. Peter Webb's talk last sem in intro to research

· Stratification: $Gr_G = \coprod_{\lambda \in X_*(T)^+} G(\mathcal{O})\lambda(\pi) G(\mathcal{O}) / G(\mathcal{O})$

Let $Gr_\lambda = G(\mathcal{O})\lambda(\pi) G(\mathcal{O}) / G(\mathcal{O})$

Then $\overline{Gr_\lambda} = Gr_{\leq \lambda} = \bigcup_{\mu \leq \lambda} Gr_\mu$.

where $\mu \leq \lambda$ if $\lambda - \mu \in \sum_{\alpha \text{ simple } \cancel{\text{coroot}}} \mathbb{Z}_{\geq 0} \alpha$

<u>Optional</u>:

SL_2, coroots are of length 1

$Gr_{SL_2} = \bigcup_{n \in \mathbb{Z}_{\geq 0}} Gr_n$,

$\overline{Gr_n} = Gr_0 \cup Gr_1 \cup \cdots \cup Gr_n$

PGL_2, coroots are of length 2

$Gr_{PGL_2} = (\bigcup_{n \in \mathbb{Z}_{\geq 0}} Gr_{2n}) \cup (\bigcup_{n \in \mathbb{Z}_{\geq 0}} Gr_{2n+1})$

$\overline{Gr_{2n}} = Gr_0 \cup Gr_2 \cup \cdots \cup Gr_{2n}$

$\overline{Gr_{2n+1}} = Gr_1 \cup Gr_3 \cup \cdots \cup Gr_{2n+1}$

even, odd components

Constructing $\mathcal{H}$

Let $\mathcal{H} = \mathrm{Perv}_{L^+G}(Gr)$ - the category of L^+G-equivariant perverse sheaves on Gr.

- We leave perverse sheaves as kind of a blackbox
  - Essentially, they are complexes of sheaves with a dimensionality constraint, supported on some X_i. (in this case, $\mathbb{Q}_\ell$-sheaves)
  - Morphisms are derived.
  - Fact: perverse sheaves come with a "trace" tr sending A to a function ~~$\mathcal{F}$~~ $tr(A)$.
- The category is abelian and monoidal
  - monoidal structure given by $*$ = sheaf convolution

$$A_1 * A_2 = m_!(A_1 \tilde{\boxtimes} A_2)$$

some external product respecting G action

($m_! = m_*$ since ind + proper)

  - Digression - this is a hard fact that $A_1 * A_2$ is perverse and not just derived. Uses a lot of deep alg geo.
- Irreducibles $IC_\mu = IC(Gr_{\leq \mu})$

Let $\mathbb{H}^*: \mathcal{H} \to \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$ be hypercohomology (replace object by complex of injectives $I^\bullet$ take $H^*(RF(I^\bullet))$).

$\mathbb{H}^*$ is monoidal (take $*$ to $\otimes$), not quite symmetric, but this is fixable by a "sign change" (eliminating a hidden $\mathbb{Z}/2$ grading).

<u>Theorem</u>: $\check{G} = \mathrm{Aut}^{\otimes} \mathbb{H}^*$ (argument on irreducibles of ~~E~~ $\mathcal{H}$).

<u>Fact</u>: This induces the $\check{G}$ action on ~~E~~ $\mathbb{H}(A)$.

<u>Geometric ~~Sat~~ Satake equivalence</u>

$$\mathrm{Perv}_{L^+G}\, Gr \xrightarrow{\sim} \check{G}\text{-Rep}$$

given by $\mathbb{H}$ along w/ canonical action of $\check{G}$

~~moreover,~~

<u>Pf.</u> Long. Does reduce to proving the statement for tori, then making a "Satake transform" from $\mathrm{Perv}_{L^+G}\, Gr_G \to \mathrm{Perv}_{L^+T}\, Gr_T$

also works over $\mathbb{Q}_p$, but with vectors.

Conclusion

- Also works in $\mathcal{D}$-mods for $\mathbb{C}$
- Versions work for coefficients in $\mathbb{C}$, $\mathbb{F}_\ell$, $\mathbb{Z}$, but argument is tweaked.

- Gives <u>natural</u> meaning to $\check{G}$
- Hints at Langlands correspondence generally