Self-adjoint operators and zeta functions

2/3/20

Speaker: Prof. Paul Garrett (Notes by Andy Hardt)

A story:

1910's: Polya/Hilbert:

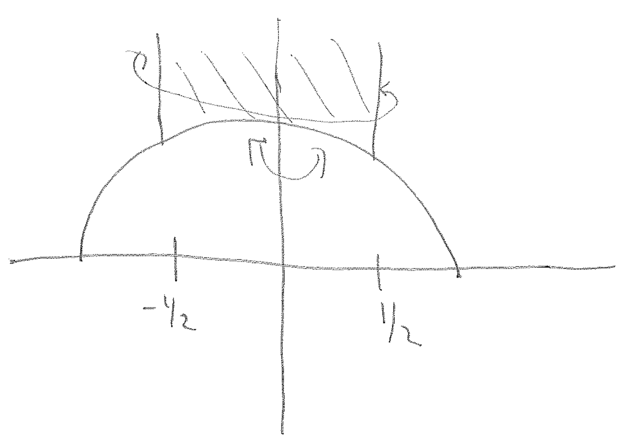
if 0's of $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

~ self-adjoint op (somewhere): e-values of $T = T^*$ are real,

then (?) could prove Riemann Hypothesis (1858-9): All 0's of $\zeta(s)$ (except $0, -2, -4, \ldots$) have $\mathrm{Re}(s) = 1/2$

$\exists$ implications . . . .

1977 Haas attempted to compute eigenvalues of invariant $\Delta = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ on upper 1/2-plane mod $SL_2(\mathbb{Z})$ (the modular curve)

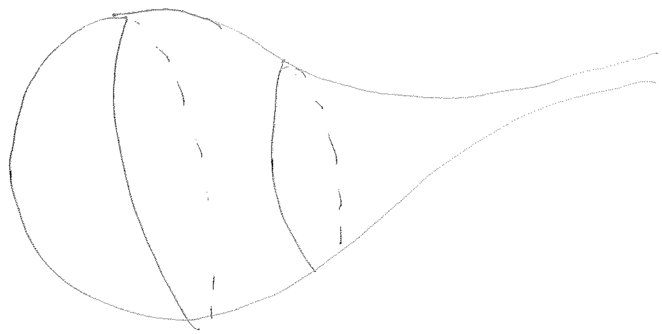


Really quotient

$\Gamma \backslash \mathfrak{H}$

Γ: $SL_2(\mathbb{Z})$

$\mathfrak{H}$: 1/2-plane



$\exists$ invt meas $\dfrac{dx\,dy}{y^2}$

True: for $\varphi, f \in C_c^\infty(\Gamma\backslash\mathfrak{H})$

$$\int_{\Gamma\backslash\mathfrak{H}} \Delta\varphi \cdot f \;=\; \int_{\Gamma\backslash\mathfrak{H}} \varphi \cdot \Delta f$$

symmetry (not quite "self-adjoint")

?!

$\langle \Delta\varphi, f\rangle = \langle \varphi, \Delta f\rangle$ "$\Delta = \Delta^*$"

?! $f \in L^2(\Gamma\backslash\mathfrak{H})$ s.t. $\Delta f \in L^2(\Gamma\backslash\mathfrak{H})$

(& if using loss of accuracy at corners?!?)

Haas sent a list of spectral parameters s s.t. (?!) $s(s-1) = \lambda_s$ is e-value of (?!) Δ

$\boxed{\text{List}}$ included (correct) e-values of "cuspforms" in $L^2(\Gamma\backslash\mathfrak{H})$

& (Hejhal, Stark ~ 1977-8)

recognized some $s=0$ of ζ → no causality

checked 0's of $L(s,\chi)$

Hejhal 1979-1981 checking

exactly 0's of ζ & $L(s,\chi)$ <u>missing</u>

($\Longleftrightarrow$ 0's of ζ & $L(s,\chi)$ are garbage) (~ Green's fun?!)

Haas really solved $\underbrace{(\Delta-\lambda)u = \cancel{0}}_{\text{not e-value eqn}}\ \delta^{afc}$ at corners & images

not e-value eqn :(

$\cancel{\Rightarrow}\ \lambda_s \in \mathbb{R}$:(

Y. Colin de Verdière

$\exists$ ~1980 precedent for magically converting inhomog to homog!

$\overset{?}{\Rightarrow}$ nevertheless, those λ_s's w/ δ <u>are</u> still e-values of self-adj. operators, so <u>real</u> ?!?!?

→ Use/redo Lax-Phillips <u>Orange</u>

scattering theory for afns in 2D

Can convert $(\Delta - \lambda_s)u = \eta_a$ to homog eqn!

$$\left(\eta_a(f) = \int_0^1 f(x+ia)\,dx\right)$$

& $\tilde{\Delta}_a = \sim \Delta$ ~ ignoring η_a (&! $\tilde{\Delta}_a^* = \tilde{\Delta}_a$ ✓)

↳ has purely discrete spectrum

In Hilbert spaces, can exist non-triv spaces not spanned by e-vectors!

CdV used to prove merom. cont. of E_s's

Δ on $\mathbb{R}$

$L^2(\mathbb{R}) \ni f(x) \, (\in S)$

$$\int_{\mathbb{R}} e^{2\pi i \xi x} \hat{f}(\xi)\, d\xi \quad \not\in L^2$$

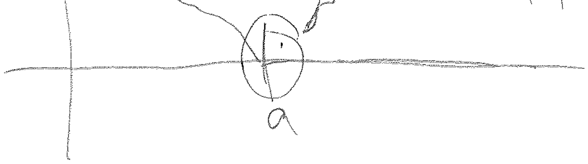
cannot be literal integral

$$L^2(\Gamma\backslash\mathfrak{h}) \ni f = \sum_{\substack{\text{cfm}\\ F}} \langle f, F\rangle F + \frac{\langle f, 1\rangle \cdot 1}{\langle 1, 1\rangle} + \frac{1}{4\pi i}\int_{\mathrm{Re}(s)=1/2} \langle f, E_s\rangle \overset{\not\in L^2}{E_s}\, ds$$

e-fun's

But for $\tilde{\Delta}_a$, cont $\underset{\text{smaller}}{\rightsquigarrow}$ discrete?!?

+ new/exotic e-funs are "truncated" Eis!



How can elliptic ops. Δ have non-smooth efuns?!

Legit: Inhomog $(\Delta - \lambda_s) u = \eta_a$

$$(\tilde{\Delta}_a - \lambda_s) u = 0 \quad \underbrace{\left(\begin{matrix} + \\ \eta_a u = 0 \end{matrix} \right)}_{\text{for self-adjoint operator}}$$

$+$ need target $\in H^{-1}$: in 2D, $\delta \in H^{-1-\varepsilon}$ $\forall \varepsilon > 0$, not $\varepsilon = 0$ $\quad \left[\delta \in H^{-\frac{\dim}{2} - \varepsilon}, \forall \varepsilon > 0 \right.$

$\tilde{\Delta}_a$ = Friedrich extn

$$L^2 \ni f = \cancel{\sum_{F \text{ cfm}} \langle f, F \rangle \cdot F} + \text{cont} + \frac{1}{4\pi i} \int \bigcirc$$

$\xrightarrow{\perp\text{-proj. in Hsp.}} 0$

asymptotics of $\zeta(s) L(s, \chi)$ on $\mathrm{Re}(s) = 1$

$\Rightarrow \delta^{\mathfrak{A}c} \in H^{-3/4 - \varepsilon}$ $\forall \varepsilon > 0$

$\subset H^{-1}$ ✓

((CdV 1982/3)

! In CdV's merom. cont.,

$(\tilde{\Delta}_a - \lambda_s) u = \eta_a$ & $\mathrm{Re}(s) \geq 1/2$, then $\eta_a u = 0$ } ignore

Lax-Phillips

? Forget $\delta^{nc} u = 0$ part of description of $\tilde{\Delta}_a$ — RH?

$(\tilde{\Delta}_\eta - \lambda)u = \delta^{nc}$ & $\delta^{nc} u = 0$ $\boxed{\text{different } \tilde{\wedge}}$

<u>Ask</u>: λ_s s.t. $(\Delta - \lambda_s)u = \delta^{nc}$ & $\delta^{nc} u = 0$?!

rel'ns, but no simultaneous

<u>Thm 0</u>: The discrete spectrum, <u>if any</u>, of $\tilde{\Delta}_{\delta^{nc}}$

is λ_s s.t. $Re(s) = 1/2$ & $\zeta(s)L(s,\chi) = 0$

Thm: At most 94% of 0's of ζ either in disc. sp. of $\tilde{\Delta}_{\delta^{nc}}$ //

Assuming RH + Montgomery pair correlation

$S = x^3 \circ i\frac{\partial}{\partial x} + i\frac{\partial}{\partial x} \circ x^3$ $\quad$ $\underline{Su = \lambda u}$

$$u = \begin{cases} e^{i\lambda/4x^2} & (\text{at } x \neq 0) \\ 0 & (x = 0) \end{cases}$$

$\in L^2$ only for $Im\,\lambda < 0$

& non-real evalues ?!

S is <u>symmetric</u>, but has <u>no</u> self-adjoint ext'ns: really found e-values for $S^* \neq$ symmetric