Overview

Use analysis, geometry, etc. to understand representation theory, number theory, etc.

Along the way, discover the truth behind weird phenomena in representation theory, combinatorics, etc

e.g. Langlands correspondence

$$\left\{ \begin{matrix} n\text{-dim} \\ \text{Galois} \\ \text{rep'ns} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{irreducible,} \\ \text{cuspidal,} \\ \text{automorphic reps} \\ \text{of } GL_n(\mathbb{A}_F) \end{matrix} \right\}$$

appearance and construction of "dual" groups.
(weird exchanging laws for generalizations of above)

Setting

F - non Archimedean local field (complete DVF)
 e.g. $\mathbb{Q}_p$, $k((t))$ for $k = \mathbb{F}_q$

$\mathcal{O}$ - ring of integers   local in ring sense
 e.g. $\mathbb{Z}_p$, $k[[t]]$

$\mathfrak{m} \subset \mathcal{O}$ maximal ideal
 e.g. (p), (t)

$k = \mathcal{O}/\mathfrak{m}$ some $\mathbb{F}_q$
 e.g. $\mathbb{F}_p$, $k = \mathbb{F}_q$

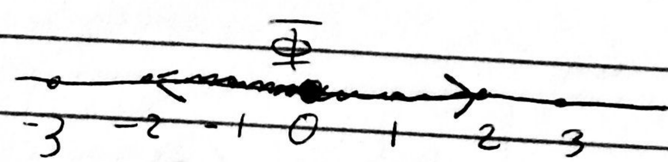
G - split, connected reductive group over F
$T \subset B$, maxtorus / Borel pair
$W = N_G(T)/T$ Weyl group

e.g. $SL_2 = \{A \in Mat_2 \mid \det A = 1\}$
$T = \{\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}\}$
$B = \{\begin{pmatrix} a & b \\ & a^{-1}\end{pmatrix}\}$

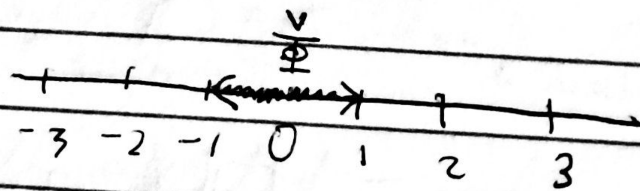
Let $X^\circ(T) = Hom(T, F^\times)$, characters
$X_\circ(T) = Hom(F^\times, T)$ cocharacters
Then composition $X^\circ(T) \times X_\circ(T) \to Aut(F^\times) \cong \mathbb{Z}$
gives a natural pairing.

Let $\Phi \subset X^\circ(T)$ roots, $\check{\Phi} \subset X_\circ T$ coroots

e.g. for SL_2, $\Phi = \{\begin{pmatrix} a & \\ & a^{-1}\end{pmatrix} \mapsto a^2, \begin{pmatrix} a & \\ & a^{-1}\end{pmatrix} \mapsto a^{-2}\}$
$\check{\Phi} = \{a \mapsto \begin{pmatrix} a & \\ & a^{-1}\end{pmatrix}, a \mapsto \begin{pmatrix} a^{-1} & \\ & a\end{pmatrix}\}$
and $X^\circ(T) \cong X_\circ(T) \cong \mathbb{Z}$
Pictorially
$X^\circ(T)$



$X_\circ(T)$



pairing multiplication
Fact: $X^\circ(T)$ controls structure of G, after
via Φ. $X_\circ T$ is something dual ?

The Spherical Hecke algebra

- F comes with a topology given by a metric. F is locally compact.
- $\mathcal{O} \subset F$ is the closed unit disk and is compact
- Hence $K = G(\mathcal{O}) \subset G(F)$ is compact, $G(F)$ locally compact.

Let's do some integration.

$$\mathcal{H}(G,K) = \left\{ f : G \to \mathbb{Z} \;\middle|\; \begin{matrix} f \text{ locally constant} \\ f \text{ compactly supported} \\ f(k_1 x k_2) = f(x) \ \forall k_i \in K, \\ x \in G \end{matrix} \right\}$$

Algebra under $*$

$$(f * g)(z) = \int_G f(x) g(x^{-1} z) \, dx$$

(dx normalized Haar measure)

($dx(K) = 1$).

Fact: $G(F) = \coprod_{\lambda \in X_*(T)^+} G(\mathcal{O}) \lambda(\pi) G(\mathcal{O})$ ← positive Weyl chamber

π uniformizer (e.g. $p \in \mathbb{Z}_p$, $t \in k[[t]]$)

Thus $\underset{\text{over } \mathbb{Z}}{\mathcal{H}(G,K)} = \langle \overset{c_\lambda}{\text{char}}_{K\lambda(\pi)K} \mid \lambda \in X_*(T)^+ \rangle$

This is unexpected - why cocharacters?

Fact $\mathcal{H}(G,K)$ is commutative b/c (G,K) Gelfand pair (anti-involution exists, fixing T.

By a quick calculation on # cosets, $\mathcal{H}(T, T(\mathcal{O})) \cong \mathbb{Z}[X_*(T)]$, by $c_\lambda \longleftrightarrow \lambda$.

The Satake Transform

Fix the Haar measure on $N = \{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \}$

~~dx~~ dn with $dn(N(\mathcal{O})) = 1$,

and $\delta : B \to \mathbb{R}_+^\times$ given by $d(bnb^{-1}) = \delta(b)\,dn$.

the modular function, trivial on N.

(δ is a character of T).

Define $\mathcal{S} : \mathcal{H}(G,K) \to \mathcal{H}(T, T(\mathcal{O})) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm\frac{1}{2}}]$

by

$$\mathcal{S}f(t) = \delta(t)^{\frac{1}{2}} \int_N f(tn)\,dn$$

($q^{\pm\frac{1}{2}}$ comes from $\delta^{\frac{1}{2}}$)

$\mathcal{S}$ is an injection. Furthermore

<u>Theorem</u> (Satake iso). The image of $\mathcal{S}$ lies in $(\mathcal{H}(T, T(\mathcal{O})) \otimes \mathbb{Z}[q^{\pm\frac{1}{2}}])^W \cong$

$(\mathbb{Z}[X_\bullet T] \otimes \mathbb{Z}[q^{\pm\frac{1}{2}}])^W$. ~~The map is~~

Recalling that ~~$(\mathbb{Z}[X_\bullet T] \otimes \mathbb{Z}[q^{\pm\frac{1}{2}}])^W$~~

$(X_\bullet T)^W \cong$ ~~X~~ $X_\bullet T^+ \cong X^\bullet \check{T}^+$, we get

an iso.

$$\mathcal{S} : \mathcal{H}(G,K) \xrightarrow{\sim} R(\check{G}) \otimes \mathbb{Z}[q^{\pm\frac{1}{2}}].$$

where $R(\check{G})$ is the representation ring of the dual group (that with dual root data).

e.g. $\mathcal{H}(SL_2(F), SL_2(\mathcal{O})) \cong R(PGL_2(\mathbb{C})) \otimes \mathbb{Z}[q^{\pm\frac{1}{2}}]$

So, the spherical Hecke algebra encodes info on the representations of the <u>dual</u>.

Common dual pairs

GL_n	SL_n	SO_{2n+1}	SO_{2n}	E_8
GL_n	PGL_n	Sp_{2n}	SO_{2n}	E_8

"swaps root lengths".

<u>Proof idea</u>

If $\lambda, \mu \in P^+$, we can find $S(c_\lambda)(t)$ for $t = \mu(\pi)$. Each $x_i = t(x_i)\, n(x_i) \in TN = B$

Then $$S(c_\lambda)(t) = \delta^{\frac{1}{2}}(t) \int_N c_\lambda(tn)\, dn$$

$$= \underset{\text{property of } \delta}{q^{-\langle \mu, \rho \rangle}} \sum_i \underbrace{\int_{N \cap t^{-1} x_i K} dn}_{=1}$$

$$= q^{-\langle \mu, \rho\rangle} \left| \{ i : t(x_i) \equiv \mu(\pi) \mod T(A) \} \right|$$

$\Rightarrow$ $S(c_\lambda)$ counts diagonal entries with valuation = to that of $\mu(\pi)$.

Sample calc: $\mathcal{S}(c_\lambda)(\lambda(\pi)) = q^{\langle \lambda, \rho \rangle}$

since $\mathcal{S}(c_\lambda)(\mu(\pi)) \neq 0 \Rightarrow \mu \leq \lambda$,

$$\mathcal{S}(c_\lambda) = q^{\langle \lambda, \rho \rangle} \chi_\lambda + \sum_{\mu < \lambda} a_\lambda(\mu) \chi_\mu .$$

Applications

- Representation theory of $\mathcal{H}(G, K)$ ~~tm~~ influences that of $R(\check{G})$.

- Useful in L-functions

Characters $\omega : R(\check{G}) \otimes \mathbb{C} \to \mathbb{C}$

$\longleftrightarrow$ ssimp conjugacy classes s in $\check{G}(\mathbb{C})$.

Then $\omega_s(\chi_\lambda) = \chi_\lambda(s) = tr(s|_{V_\lambda})$.

Then $\mathcal{S} : \mathcal{H}(G, G(\mathcal{O})) \otimes \mathbb{C} \to R(\check{G}) \otimes \mathbb{C}$

gives a parametrization