YANG BAXTER EQUATIONS FOR METAPLECTIC ICE

- thank everyone, and specifically thank committee.

today: Whittaker fcns and Metaplectic Ice

3 sections: $\boxed{1}$ Whittaker fcns (& w/ impt)

$\boxed{2}$ Metaplectic Ice (explicit expression for Whit fcns)

$\boxed{3}$ Quantum Connections (writing Whit fcns in terms of Met Ice leads to surprising connection) special val of whit. fcns ⟶ facts about q-gps

Part $\boxed{1}$: Whittaker fcns

general: automorphic representations ⟶ (nice analytic) generating fcns (ex: L-fcns)

numeric invariants ↖ assoc. to special fcns

ex: Schur fcn, arise as trace of character

Whittaker fcn ↖ talk about today.

a nice illustrative case: $G = GL_2(F)$, F nonarch local field

$T = \left\{ t^\lambda := \begin{pmatrix} u\varpi^\lambda & \\ & u^{-1}\varpi^{-\lambda} \end{pmatrix} \text{ for } \lambda \in \mathbb{Z} \right\}$ u unit $\in \mathcal{O}^\times$

one param subgp; ϖ uniformizer

to construct principal series representations,

- start with unram char on torus precisely $\chi_z : t^\lambda \longmapsto z^\lambda$ $z \in \mathbb{C}$
- inflate trivially to Borel subgp $B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$
- induce up to G : $(\pi_z, I(\chi_z)) := \mathrm{Ind}_B^G(\chi_z)$. ↲ inf dim'l set of fcns transform by χ_z on left

We then define a particular fcn on this representation:

- first: for K max'l cpt, $I(\chi)^K$ is 1-dim; fix one ϕ_K = spherical vector
- fix unram character ψ of F.
- Def: $\boxed{\text{Whittaker fcnl}}$ on $(\pi_z, I(\chi_z))$ is linear fcnl W s.t.

$$W\left(\pi\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} v\right) = \psi(x)\, W(v)$$

↑ elts of unipotent

↑ space of such is 1-dim; fix one: W_z

- Def: $\boxed{\text{spherical Whittaker fcn}}$ is eval of W_z against ϕ_K.

$$\mathcal{W}_z(g) := W_z(\pi(g)\,\phi_K).$$

- suffices to calculate on torus elements, for which

$$W_z(t^\lambda) = \begin{cases} (*) \cdot S_\lambda(z) & \lambda \text{ dominant} \\ & \text{else} \end{cases}$$

indep of λ ; Schur polynomial (write that) impt symmetric poly in z

due to Shintani and later Casselman-Shalika more generally ← show Schur SLr

- not hard to show 0, harder to get form, As!

COOL THINGS: 1) explicit formula is good, esp. for Rankin-Selberg, Langlands-Shahidi calculations

2) mx coeffs of inf dim rep ⇝ f-dim poly (discrete / cts flip)

3) $S_\lambda(z)$ is char of highest weight rep of $PGL_2(\mathbb{C})$.

For arb. reductive gp G, replace $PGL_2(\mathbb{C})$ with Langlands dual group ${}^L G(\mathbb{C})$

- one of first instances where dual group appeared, was in Langlands' calculations of this sort.

metaplectic complications

how much of this extends to ~~other groups?~~ metaplectic case?

Def: $\boxed{\text{metaplectic covering group}}$ is central extn of red gp G by SES $1 \longrightarrow \mu_n \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$ n s.t. $\mu_{2n} \subset F$.

- for us, $G = GL_r$, so most relevant source is 1984 Kazhdan-Patterson, also Weil, Matsumoto, Brylinski-Deligne, & McNamara

What does $\tilde{G}$ look like? $G \times \mu_n$ as set

$$(g_1, \zeta_1)\cdot(g_2,\zeta_2) = (g_1 g_2, \underbrace{\sigma(g_1,g_2)}_{\text{cocycle in } H^2(G,\mu_n)} \zeta_1\zeta_2)$$

In this paper, although not so much in talk, we treat covers using the parametrization considered by Brylinski-Deligne, where cocycle ⟶ bilinear form on cochar of torus

under this treatment $\{\text{met covers of } GL_r\} \longleftrightarrow \{B = m\cdot I,\ m \in \mathbb{Z}/n\mathbb{Z}\}$

Problems in Extending Earlier Construction: 1) constructing principal series reps $(\pi_\chi, I(\chi))$

↳ preimage $\tilde{T}$ of torus no longer abelian, but we want to ~~ind~~ start w/ unram char on ab subgp

↳ can go into more detail if questions

2) constructing Whittaker fcns

↳ $\tilde{T}$ not abelian ⇒ space of Whittaker fcnls no longer 1-dim, but controlled by index of max'l ab subgp in $\tilde{T}$, so finite-dim

3) computing whittaker fcns

- need to understand relationship between Whittaker fcnls and intertwining operators $A_s : I(\chi) \longrightarrow I(s\chi)$, which permute complex param z.

- boils down to rank 1 calculation, 1984 K-P presented as scattering matrix for $\widetilde{GL_r}$
  - 2013 Chinta Offen ramped up into CS
  - 2016 McNamara cleverly applied to covers of arbitrary reductive gps

BACK TO COOL THINGS:
1) still applies, have explicit formula ← but it's messy...
2) do we have a similar interpretation of spherical Whittaker fcn as linear fcnl on a nice algebraic object?

Having asked that, let's turn to part 2 for something seemingly completely different

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Part $\boxed{2}$ Metaplectic Ice

- Recently, Brubaker, Buciumas, & Bump proved connections between metaplectic Whittaker fcns for ~~the~~ a specific n-fold cover of GL_r, specifically the one $B=I$. and ~~solvable~~ six-vertex lattice models techniques from statistical mechanics
- My paper extends this to any metaplectic cover of GL_r.

Construction

* ROUGH THM *

To make Metaplectic Ice
- draw finite 2-D grid of vertices
- connect w/ edges
  - each edge ~~gets~~ will get a $\pm$ spin :
- number rows bottom to top, starting at 1
- number columns right to left, starting at 0

- given torus element t^λ for λ a dominant weight, we can define boundary conditions on grid of right size : need ~~#~~ r rows and at least $\lambda_1 + r$ columns

$\boxed{\text{system}}$ $\mathfrak{S}_\lambda$
- $+$ along left & bottom
- $-$ along right
- on top : $-$ on columns in $(\lambda+\rho)$, where $\rho = (r-1, \ldots, 2, 1, 0)$.

Ex: $\lambda = (2,1,0)$ $\Rightarrow (\lambda+\rho) = (4,2,0)$.
$\rho = (2,1,0)$

- fill in ~~edges~~ so that each vert~~ex~~ looks like one of the 6 admissible vertices
  → this is called an admissible $\boxed{\text{state}}$
- Def: $\boxed{\text{charge}}$ = # + on or to right of horizontal edge
  ↳ label edges with their charge
- Def: ~~state~~ fix $n_Q \in \mathbb{Z}_+$, state is $\boxed{n_Q\text{-admissible}}$ if every horizontal edge w/ $-$ spin has charge $\equiv 0 \pmod{n_Q}$.

- mention charge mod n_Q.

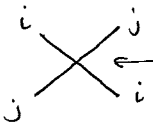
Def: $\boxed{\text{partition function}}$: attach weights to vertices

- given a state, Boltzmann wt is product of wts of vertices in that state
- given a system, partition fcn is sum of Boltzmann wts for states w/ those bdry conditions
  ($\shortparallel$ bdry conds)
  $Z(\mathfrak{S}_\lambda)$.

Thm [BBB B=I, F general] Up to normalization, ~~the~~ $Z(\mathfrak{S}_\lambda)$ is a value of a spherical Whittaker fcn on an n_Q-fold cover of GL_r.

How do we see intertwining operators here? The z-param shows up in the z_i's in our wts, so acting by an intertwining op corresp. to switching two rows

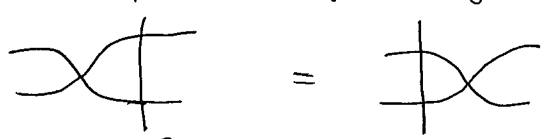
$\Rightarrow$ need diagonal ~~ve~~ vertices



can choose set of wts $R_{z_i z_j}$ s.t. our ice model satisfies certain Yang Baxter eqns that match relations on intertwining ops.

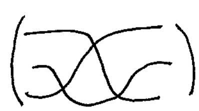
YBEs

Thm [BBB, F] fixing bdry cond, the partition fcns of the following two systems are equal:



- there's a particularly nice example done out on pg. — of my paper, which shows how the Gauss sums interact here

Also have

Thm [BBB, F] Z () = Z ()

Thm [BBB, F] Z () $= \begin{cases} 1 & \alpha=\gamma,\ \beta=\delta \\ 0 & \text{else} \end{cases}$

Could prove these using brute force: only 32 cases for bdry spins for YBE 1, and this is how BBB did it (see Appendix for my modified calcs), but want to be more clever.

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rt $\boxed{3}$ Connections

Seems reasonable to look at ~~other~~ natural sources of solutions to YBEs to see if our model is related to them; BBB saw that their model was connected to quantum gps, and my paper shows that mine is too.

In particular, we look at affine quantum gp

$$\mathcal{U} = \mathcal{U}_{\sqrt{v}}(\widehat{\mathfrak{gl}}(1|n_Q)) \leftarrow \text{quasitriangular Hopf superalgebra}$$

it gets all the adjectives and all the decorations

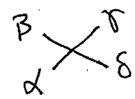
↳ to unpack: quantum gp ≈ generalization of universal enveloping algebra

- affine (↖ hat part) ~ Lie alg is central extension by complex torus
- super $(1|n_Q)$ part ~ $\mathbb{Z}/2\mathbb{Z}$ graded generalization of Lie alg
  - $1 \Leftrightarrow 0$ graded part
  - $n_Q \Leftrightarrow +$ graded part

- for every $z \in \mathbb{C}^\times$, there is a $(1|n_Q)$-dim eval $\mathcal{U}$mod V_z corresp to std rep
  - we assoc our spins $-0, +0, \ldots, +n_Q-1$ w/ basis elts of this module, allows us to rep these mods w/ horizontal strands
- q-gp comes equipped w/ univ. R-mx, which measures failure of $V \otimes W$ to be isom to $W \otimes V$.

Express as endom on $V_{z_i} \otimes V_{z_j}$ as

$$R_{ij}(v_\alpha \otimes v_\beta) = \sum_{\gamma,\delta} R^{\alpha,\beta}_{\gamma,\delta}(v_\gamma \otimes v_\delta)$$

and compare $R^{\alpha,\beta}_{\gamma,\delta} \rightsquigarrow$ wt for 

aaalmost the same.

- Out of box R-mx doesn't have Gauss sums, which we need
  ↳ apply procedure called $\boxed{\text{Drinfeld twist}}$
  - doesn't change underlying vsp of q-gp
  - does tweak R-mx
  - preserves YBEs.
- $\mathcal{U}$'s R-mx satisfied YBE 2 & 3, so our model does too!
- note: no quantum interpretation of YBE 1, b/c would need 2-dim $\mathcal{U}$-mod (b/c ± spins), and this $\mathcal{U}$ doesn't have one.

Brings us back to earlier question: do we have interpretation of Whit as lin fcn on nice alg object? Yes.

Specifically $\mathcal{U}_+ = \mathcal{U}_{\sqrt{v}}(\widehat{\mathfrak{gl}}(n_Q)) \leftarrow$ just pos graded part

Thm [BBB, F] scattering matrix arises as R-mx of Drinfeld twist of $\mathcal{U}_+$

So we can interpret spherical Whittaker fcn as the partition fcn on this quantum group.