11/25. Crystalline cohomology and Katz's conjecture.

[Ref]: Pierre Berthlot. Arthur Ogus: Notes on Crystalline cohomology

[Ref]: A. Grothendieck: Crystals and the De Rham Cohomology of schemes.

~~§I. 1. Int~~ 1. Why crystalline cohomology 2. What is ...

3. Katz's conjecture 4. number of rational points.

§1. Why

<u>Def</u>: A Weil cohomology theory is a contravariant functor:

H^* : {Smooth projective varieties over k} (any char)

$\longrightarrow$ {graded k-algebra}. (k: char $= 0$)

$H^*(X)$.

s.t. satiesfies axioms:

- Finiteness
- Vanishing property
- Poincaré duality
- Künneth isomophism.
- Cycle map
- Weak and Strong Lefschetz

<u>e.g.</u>: 1. Singular coho 2. Algebraic de Rham coho.

3. Étale coho (l-adic). 4. Crystalline coho (p-adic).

- Zeta function of a variety over a finite field:

~~X_0 pro~~ X proper smooth over $k = \mathbb{F}_q$. $N_s = \# X(\mathbb{F}_{q^s})$. $\dim X = d$.

$$Z(T; X/k) = \exp\left(\sum_{i=1}^{\infty} \frac{N_s T^s}{s}\right) \overset{\text{fixed pt. seen as Fr}}{=} \prod_{i=0}^{2d} P_i(T)^{(-1)^{i+1}} \quad P_i(T) = \det(1 - T \cdot Fr^*|_{H^i_{et}(\bar{X}, \mathbb{Q}_l)})$$

Grothendieck trace formula & Lefschetz fixed point formular (the reason of étale).

where $l \neq \operatorname{char} q$. $\bar{X} = X \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. $\boxed{Fr}$ is the relative Frobenious.

<u>Weil conjecture</u>: $P_i(T) \in \mathbb{Z}[T]$, the eigenvalues $\{\alpha_{ij}\}$ of $Fr^*|_{H^i}$ (i.e. the reciprocal roots of $P_i(T)$) has complex <u>absolute value</u> $|\alpha| = q^{\frac{i}{2}}$.

Take log deri. $N_r = \sum_{0 \le i \le 2d} (-1)^i \sum_{j=1}^{\beta_i} \alpha_{i,j}^r = q^{rd} + 1 + \sum_{1 \le i \le 2d-1} (-1)^i \sum_{j=1}^{\beta_i} \alpha_{i,j}^r$. ($\beta_i$ = $\dim H^i$ (Betti number))

<u>Cor</u>: $N_r = q^{rd} + 1 + O(q^{r(d-\frac{1}{2})})$

- l-adic value: α is a e.g. of H^i. then $\frac{q^d}{\alpha}$ is a e.g. of H^{2d-i} (Poincaré duality).

α alg. int $\Rightarrow$ α: l-adit unit.

- p-adic value? (a p-adic cohomology theory).

§ Crystalline coho: ~~l-adic~~ étale. $H^*_{et}(X, \mathbb{Z}_l) := \varprojlim H^*_{et}(X, \mathbb{Z}/l^n)$.

l-adic ($\simeq \varprojlim$ l.c.c. sheaves).

l-adic, p-adic topo are not compatible, exactly taking (local systems.)

(Zariski topology is too coarse for local systems, so we need (étale site)).

• Ag de Rham: $\mathbb{H}^*(X, \Omega^k_{X/k})$. (tangent bundle. Zariski open is enough).

• Crystalline: Zariski. all thinkening ~~all orders diff~~ the.

Let k: perfect field of char $k = p$. $W = W(k)$ Witt vector. ~~$W_n = W(k)/p^n W(k)$~~

W_n: n-th Witt vector of k (e.g. $k = \mathbb{F}_p$. $W(k) = \mathbb{Z}_p$. $W_n (\mathbb{Z}/p^n\mathbb{Z})$). For a scheme X over k. define a site $\mathrm{cris}(X/W_n)$. (X not over W_n!)

<u>Obj</u>: (U, T). $U \underset{\text{Zariski open}}{\subseteq} X$. T: W_n-scheme. $U \overset{cl.}{\hookrightarrow} T$ with defining ideal of U in T nilpotent and has a <u>PD-structure</u> compatible with PD-struc on W_n.

<u>Mor</u>:
$$\begin{array}{ccc} U & \hookrightarrow & T \\ \wr\, op & & \downarrow \text{PD-mor.} \\ U' & \hookrightarrow & T' \end{array}$$

<u>Covering</u>: $\{(U_i, T_i)\}$ is a covering of (U, T) if $T_i \to T$ open immersion and $T = \bigcup T_i$

<u>PD-structure</u> (Devided Power Structure): A is a comm ring. I ideal in A. d.p on I are a collection of maps $\gamma_i : I \to A$ s.t. (5 axioms) ($\gamma_i(x+y) = \cdots$ $\gamma_i(xy) = \cdots$ $\gamma_{i+j}(x) = \cdots$) abstract from: if A is a $\mathbb{Q}$-algebra. let $\gamma_n(x) = \frac{x^n}{n!}$

<u>Why PD-struc</u>? (de Rham of formal lifting).

$k = \mathbb{C}$: $0 \to \mathbb{C}[[t]] \xrightarrow{d} \mathbb{C}[[t]]dt \to 0$, $\frac{t^{n+1}}{n+1} \leftarrow t^n dt$

$k = \mathbb{F}_p$, $\rightsquigarrow \mathbb{Z}_p$: $0 \to \mathbb{Z}_p\{t\} \to \mathbb{Z}_p\{t\}dt \to 0$. $\frac{t^p}{p}dt \leftarrow t^{p-1}dt$ not exist in $\mathbb{Z}_p$. H^0 not f.d!

Extra PD-struc allows us to do ~~diff~~ der and integral. $\boxed{\text{PD-structure in } W_n}$

Site $\mathrm{crys}(X/W_n) \rightsquigarrow$ Topos $(X/W_n)_{\mathrm{cris}}$. Let $\mathcal{F} \in (X/W_n)_{\mathrm{cris}}$. (e.g. $\mathcal{O}_{X/W}$. $\mathcal{O}_{X/W}((U,T)) = \mathcal{O}_T(T)$), $\Gamma(\mathcal{F}) := \Gamma(e, \mathcal{F}) := \mathrm{Mor}(e, \mathcal{F})$. e is the final object in $(X/W_n)_{\mathrm{cris}}$ (not representable). enough inj objects on sheaves of abgrp. $H^i_{\mathrm{cris}}(X/W, \mathcal{F}) :=$ i-th derived functor of $\Gamma(\mathcal{F})$.

If X sm. proper $/k$. $H^*_{\mathrm{cris}}(X/W) := H^*_{\mathrm{cris}}(X/W, \mathcal{O}_{X/W}) = \varprojlim H^*_{\mathrm{cris}}(X/W_n, \mathcal{O}_{X/W_n})$.

<u>Theorem</u>: If Y/W is a smooth lifting of X/k, then $\exists$
$$H^*_{\mathrm{cris}}(X/W) \cong H^i_{DR}(Y/W). \quad \text{(independent to lifting)}.$$

<u>Theorem</u>: $H^*_{\mathrm{cris}}(X/W) \otimes_W \underbrace{\mathrm{Quot}(W(k))}_{K}$ is a Weil coho theory. and

$P_i(T) = \det(1 - T\,\mathrm{Fr}^* |_{H^i_{et}}) = \det(1 - T\,\mathrm{Fr}^* |_{H^i_{\mathrm{cris}}})$; $\forall i$.

Katz's conjecture:

Frob on $W(k)$.

$\phi = F_r^* : H^*_{cris}(X/W) \to H^*_{cris}(X/W)$ is a σ-linear map ($\phi(ax) = a^p\phi(x)$).
is an isogeny, i.e. $\phi \otimes k$ is bijective.

$\forall m$, $H^m := H^m_{cris}(X/W)/(\text{Torsion})$ is free of finite rank over W.

Theorem (Dieudonné–Manin). Fix any m.

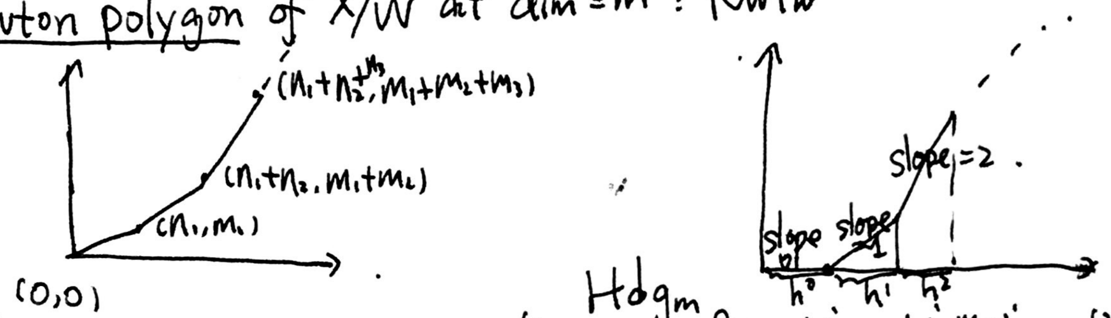
$$F_r^* \circlearrowright H^m \otimes_W W(\bar k) \simeq \bigoplus_{i=1}^{t} W(\bar k)[T]/T^{m_i} - p^{n_i}, \quad \left(\frac{n_i}{m_i} < \frac{n_{i+1}}{m_{i+1}}\right).$$

$T \curvearrowright$ mult.

where $m_i, n_i \in \mathbb{Z}_{\geq 0}$, $(m_i, n_i) = 1$, the rational number $\frac{n_i}{m_i}$ are called slopes, with multiplicity m_i.

e.g.: If $k = \mathbb{F}_p$, $\phi : H^m \to H^m$ has e.g. $\{\alpha_{m,j}\}$ with $\mathrm{ord}_p\, \alpha_{m,j}$ precisely the slopes with mult.

Newton polygon of X/W at dim $= m$: Nwt_m



Hdg_m

Hodge polygon of X/W at dim $= m$: Let $h^i := h^{i,m-i} = \dim_k H^{m-i}(X, \Omega^i_{X/k})$.

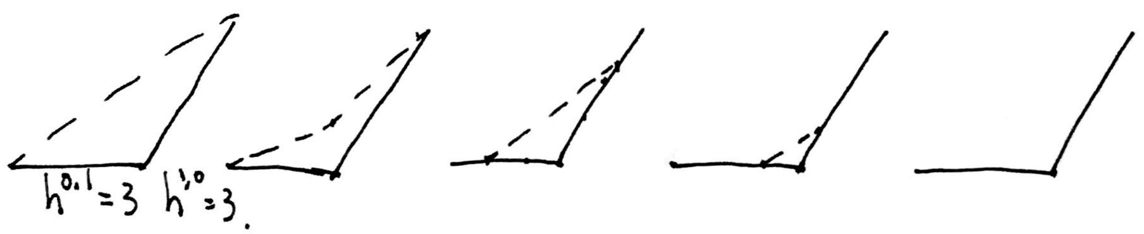
Theorem (Katz's conjecture):

If X/k is smooth and proper, then the Newton's polygon lies on or above the Hodge polygon of X/k. Moreover, assume $H^*_{cris}(X/W)$ is torsion-free and Hodge to de Rham spectral $E_1^{ij} = H^j(X, \Omega^i_{X/k}) \Rightarrow H^{i+j}_{DR}(X/k)$ degenerates at E_1, then $\cancel{H^{i,m}}$ $h^{i,m-i}$ is the ~~mult of~~ multiplicity of the elementry divisor p^i of the W-linear map $\tilde\phi : \sigma^* H^m(X/W) \to H^m(X/W)$ defined by ϕ. Also, Nwt_m and Hdg_m have the same end point (b_m, C_m), $b_m = \mathrm{rk}\, H^m_{cris}(X/W)$, $C_m =$ ~~the~~ length $H^m_{cris}(X/W)/\mathrm{Im}\,\tilde\phi$. (if X is proj, $C_m = m\frac{b_m}{2}$ by hard lefschetz)

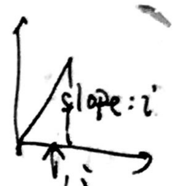
$H^{d-r} \to H^{d+r}(r) \xrightarrow{\sim} H^{d-r}(d-r)$

$\alpha \mapsto g^{d-r}\alpha$. $\quad \alpha \quad a \mapsto m-a$.

e.g.: $X/\mathbb{F}_p$: a curve of genus 3.



<u>Cor</u>: If $c = \min\{i \in \mathbb{Z}_{\geq 0} \mid h^{i,m-i} \neq 0\}$. $\forall i$. we have $\operatorname{ord} \alpha_{m,i} \geq c$



<u>Cor</u>: Assume $k = \mathbb{F}_q$. X/k smooth complete intersection of dim d with multidegree $\underline{a} = (a_1, \ldots, a_r)$ in $\mathbb{P}_k^{d+r}$. then.

$$Z(T, X/k) / Z(\cancel{\mathbb{P}_k^d/k}\, T, \mathbb{P}_k^d/k) \in \mathbb{Z}[[q^c t]]$$ (weak Lefschetz + Poincaré duality) (only $P_d(T)$ still there).

or equivalently: $N_s(X/k) = N_s(\mathbb{P}_k^d/k) \mod q^{cs}$.

(Deligne. SGA7. Gives a formula of c depend on $\underline{a}$).