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Introduction to Rankin-Selberg Method

- History
- Basic example (holomorphic modular forms for $SL(2,\mathbb{Z})$)
- A more general case : $GL_m \times GL_n$
- Application
- Reference.

History

Rankin - Selberg Method

Rankin → Robert Alexander Rankin

Selberg → Atle Selberg

is introduced independently in 1939 by Rankin, 1940 by Selberg.

This is also known as the theory of integral representations of L-functions. It is a technique for directly constructing and analytically continuing several important examples of automorphic L-functions.

- Standard L-function on GL_n (Godement-Jacquet)
- Standard L-function on classical groups (Piateski-Shapiro-Rallis)
- Standard L-function on $GL_n \times GL_m$. (Jacquet, PS, Shalika
- Tensor product L-function on $GL_n \times GL_m$. completed by Moeglin-Waldspurger)

(& was "reverse-engineered" to establish the "converse theorem")

- Exterior square on GL_n (Jacquet-Shalika, Bump-Ginzburg

...

$SL(2,\mathbb{Z})$ setting) - Basic Rankin-Selberg

Let f, g be two holomorphic cusp forms on the upper half plane $\mathfrak{h}$

cusp form : a modular form which vanishes at the cusps.

cusp form : a modular form with $c(0)=0$. ($c(0)$ is the value of f at $i\infty \to$ cusp

$f(z) = \sum_{n=0}^{\infty} c(n) e^{2\pi i n z}$

modular form of weight k:

1) holomorphic in the upper half-plane $\mathfrak{h} = \{\mathrm{Im}(z) > 0\}$

2) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

of weight k for $SL(2,\mathbb{Z})$, with Fourier expansion

$$f(z) = \sum_{n>0} a_n e^{2\pi i n z}$$

$$g(z) = \sum_{n>0} b_n e^{2\pi i n z}$$

Write $\Gamma = SL(2,\mathbb{Z})$, the modular group

Let $P = \left\{ \begin{pmatrix} * & * \\ & * \end{pmatrix} \in \Gamma \right\}$ upper triangular group.

Then we can define Eisenstein series E_s as

$$E_s(z) = \sum_{\gamma \in P \backslash \Gamma} \mathrm{Im}(\gamma z)^s$$

Here γ acts on z by $\overset{\gamma}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} z = \frac{az+b}{cz+d}$, $\mathrm{Im}(\gamma z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$

These are our focus on analytic properties of L-func too.

- One can verify : E_s converges absolutely for $\mathrm{Re}(z) > 1$
- E_s is $SL(2,\mathbb{Z})$-invariant.
- E_s has an analytic continuation to $s \in \mathbb{C}$.
- E_s has a functional equation :

zeta func : $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$

let $\xi(s) = \pi^{-s/2} \Gamma(\frac{s}{2}) \zeta(s)$ w/ gamma factor $\pi^{-s/2}\Gamma(s/2)$

Functional equation $\xi(2s) E_s = \xi(2-2s) E_{1-s}$.

Peterssen inner product for weight k modular forms:

$$\langle f, g \rangle = \int_{\Gamma\backslash\mathfrak{h}} \underbrace{f(z)\,\overline{g(z)}\, y^k}_{\Gamma\text{-invariant}} \underbrace{\frac{dx\,dy}{y^2}}_{SL(2,\mathbb{R})\text{-inv. measure on}}$$

Rankin-Selberg integral

$$\langle f\cdot E_s, g\rangle := \int_{\Gamma\backslash\mathfrak{h}} f(z)\,\overline{g(z)}\,E(z,s)\,y^k\,\frac{dx\,dy}{y^2}$$

- converges for all $s\in\mathbb{C}$ (if one of the forms is cuspidal)
- analytic continuation

Thm: $\langle f\cdot E_s, g\rangle = (4\pi)^{-(s+k-1)}\,\Gamma(s+k-1)\sum_{n\geq 1}\frac{a_n\overline{b_n}}{n^{s+k-1}}$

$\xi(2s)\langle f\cdot E_s, g\rangle = (4\pi)^{-(s+k-1)}\,\xi(2s)\,\Gamma(s+k-1)\sum_{n\geq 1}\frac{a_n\overline{b_n}}{n^{s+k-1}}$

has an analytic continuation to $\mathbb{C}$ w/ poles at most at $s=0,1$.

Pf: For an integral P-inv. function φ on $\mathfrak{h}$, we have an identity similar to Fubini's thm:

(Haar integral?)
$$\int_{P\backslash\mathfrak{h}} \varphi(z)\,\frac{dx\,dy}{y^2} = \int_{\Gamma\backslash\mathfrak{h}} \sum_{\gamma\in P\backslash\Gamma} \varphi(\gamma z)\,\frac{dx\,dy}{y^2}$$

Let $\varphi(z) = y^s\cdot f(z)\,\overline{g(z)}\,y^k$

$$\Rightarrow \int_{P\backslash\mathfrak{h}} y^s f(z)\,\overline{g(z)}\,y^k\,\frac{dx\,dy}{y^2} = \langle f\cdot E_s, g\rangle$$

For fundamental domain for $P\backslash\mathfrak{h}$, take

$$\Phi = \{z = x+iy \in \mathfrak{h} : 0\leq x\leq 1\}$$

$$\int_{P\backslash\mathfrak{h}} y^s f(z)\,\overline{g(z)}\,y^k\,\frac{dx\,dy}{y^2} = \sum_{m,n\geq 1} a_m\overline{b_n}\int_{y>0} y^{s+k-1}\Big(\underbrace{\int_{0\leq x\leq 1} e^{2\pi i(m-n)x}\,dx}_{\{0,\,1\}}\Big)$$

$$= \sum_{n\geq 1} a_n\overline{b_n}\int_{y>0} y^{s+k-1} e^{-4\pi n y}\,\frac{dy}{y}$$

$ny\to y$
$$= (4\pi)^{-(s+k-1)}\sum_{n\geq 1} a_n\overline{b_n}\,n^{-(s+k-1)}\int_{y>0} y^{s+k-1} e$$

$$= (4\pi)^{-(s+k-1)}\,\Gamma(s+2k-1)\sum_{n\geq 1}\frac{a_n\overline{b_n}}{n^{-(s+k-1)}}.$$

This is true for $\mathrm{Re}(s) > 1$,
And by the identity principle of Eisenstein series (analytically continuous), it is absolutely convergent for $s\in\mathbb{C}$, away from

Rankin's original purpose in considering the tensor product L-func. was to ~~approach Ramanujan's~~? conjecture on the size of Hecke ~~eigenvalues~~.

the poles of the Eisenstein series (b/c the Eisenstein series is of moderate growth, and the cusp forms are of rapidly decay.) □

- Functional Equation.

(Cogdell's paper) (Jacquet, PS, Shalika)

$GL_n \times GL_m$ convolution

different between $n=m$ and $n \neq m$

$\uparrow$ involve with Eisenstein series.

$\boxed{n=m}$

F: global field, $\mathbb{A}$: adele ring,

P: standard maximal parabolic subgroup of GL_n with levi factors

? $GL_{n-1} \times GL_1$ $\left(\frac{GL_{n-1} \ \ *}{\ \ \ GL_1}\right)$

$\delta: P(\mathbb{A}) \longrightarrow \mathbb{C}^*$ modular quasi-character

$$\delta\begin{pmatrix} h & * \\ & a \end{pmatrix} = |\det h|\, a^{-(n-1)}, \quad h \in GL(n-1, \mathbb{A}), \ a \in \mathbb{A}^*.$$

For $s \in \mathbb{C}$, let $f_s \in \mathrm{Ind}_{P(\mathbb{A})}^{GL(n,\mathbb{A})}(\delta_P^s)$

- f_s smooth func.
- $f(pg) = \delta_P(p)^s \ f_s(g)$

also assume
- f_s restricted to a standard max'l compact subgroup of

? $GL(n, \mathbb{A})$ is independent of s.

WLOG, we can write

$$\mathrm{Ind}_{P(\mathbb{A})}^{GL(n,\mathbb{A})}(\delta_P^s) = \otimes_v' \ \mathrm{Ind}_{P_v}^{GL(n,F_v)}(\delta_P^s),$$

restricted product

$$\Rightarrow f_s(g) = \prod_v{}' f_{s,v}(g_v), \text{ where } f_{s,v} \in \mathrm{Ind}_{P_v}^{GL(n,F_v)}(\delta_P^s).$$

We have an Eisenstein series

$$E(g,s) = \zeta(ns) \sum_{P_F \backslash GL(n,F)} \delta_P(\gamma g)^s$$

- It is convergent for $Re(s)$ suff. large
- It has meromorphic continuation to all s.

Let φ_1, φ_2 be $GL(n)$ cusp forms in automorphic repn π_1, π_2.

consider

$$\int_{GL(n,F) \underset{\substack{\downarrow \\ \text{center of } GL(n)}}{Z_{\mathbb{A}}} \backslash GL(n,\mathbb{A})} \varphi_1(g)\, \varphi_2(g)\, E(g,s)\, dg .$$

After "unfolding", ||

$$\zeta(ns) \int_{N_{\mathbb{A}} Z_{\mathbb{A}} \backslash GL(n,\mathbb{A})} W_1(g)\, W_2(g)\, f_s(g)\, dg \quad \cdots \quad (*)$$

where W_1, W_2 are Whittaker functions defined as

$\psi : \mathbb{A}/F \to \mathbb{C}$ nontrivial additive character

N: algebraic subgroup of upper triangular unipotent matrices in $GL(n)$,

define a character $\psi_N : N_{\mathbb{A}} \to \mathbb{C}$ as

$$\psi_N(n) = \psi\left(\sum_{i=1}^{n-1} n_{i,i+1}\right)$$

Whittaker model

Then $W_1(g) = \int_{N_F \backslash N_{\mathbb{A}}} \varphi_1(ng)\, \psi(n)\, dn \quad \in \mathcal{W}(\pi_1, \psi)$

$W_2(g) = \int_{N_F \backslash N_{\mathbb{A}}} \varphi_2(ng)\, \psi(n)^{-1}\, dn \quad \in \mathcal{W}(\pi_2, \bar{\psi})$.

Because of the uniqueness of Whittaker model $(\dim \mathcal{W} \le 1)$ irr, adm.

these functions are Euler products, ie if we assume WLOG

$\varphi_i = \bigotimes_v \varphi_{i,v}$ ~~$\Phi_i \otimes_v \Phi_{i,v}$~~, in $\pi_i = \bigotimes \pi_{i,v}$

$$W_i(g) = \prod_v W_{i,v}(g_v)$$

$\downarrow$ Whittaker function on $GL(n, F_v)$.

$$\Rightarrow \circledast = \prod_v \zeta_v(ns) \int_{N_v Z_v \backslash GL(n,F_v)} W_{1,v}(g_v) W_{2,v}(g_v) f_{s,v}(g_v)\, dg_v$$

$\downarrow$ local Dedekind zeta func.

$$\Psi(s, W_1, W_2, f_s) \boxed{\int_{NZ \backslash GL(n,F_v)} W_1(g) W_2(g) \underbrace{f_s(g)}_{|\det g|^s} dg} \text{ — local integral :)}$$

If $m \neq n$.

$\underline{m = n-1}$: global integral is "of Hecke type".

It looks like

$$\int_{GL(n-1,F) \backslash GL(n-1,\mathbb{A})} \varphi_1\begin{pmatrix} g & \\ & 1 \end{pmatrix} \varphi_2(g) |\det g|^{s-1/2} dg$$

(w/o Eisenstein series)

Since the Rankin-Selberg integrals of Hecke type are fairly rare, many authors do not use the term Rankin-Selberg unless there is an Eisenstein series.

$\underline{m < n-1}$, eg $m = n-2$.

It looks like

$$\int_{GL(n-2,F) \backslash GL(n-2,\mathbb{A})} \int_{(\mathbb{A}/F)^{n-1}} \int_{\mathbb{A}/F} \varphi_1\begin{pmatrix} \boxed{g} & & \xi \\ & 1 & x \\ & & 1 \end{pmatrix} \varphi_2(g) |\det g|^{s-1} \psi(x)\, dx\, d\xi\, dg.$$

local integrals :

$m = n-1$ $$\Psi(s, W_1, W_2) = \int_{NZ \backslash G_m} W_1\begin{pmatrix} g & \\ & 1 \end{pmatrix} W_2(g) |\det g|^{s-1/2} dg$$

$m < n-1$ $$\Psi(s, W_1, W_2; j) = \int_{NZ \backslash G_m} \int_{M(j \times m, F)} W_1\begin{pmatrix} g & & \\ x & 1_j & \\ & & 1_{k+1} \end{pmatrix} W_2(g) |\det g|^{s - \frac{n-m}{2}} dx\, dg$$

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• Main Results

Thm: π_1, π_2 repn of GL_n, GL_m of Whittaker type.

let $W_1 \in W(\pi_1, \psi)$, $W_2 \in W(\pi_2, \bar{\psi})$, then:

(i) Each of the integrals $\Psi(s, W_1, W_2, f_s)$ $(n=m)$

& $\Psi(s, W_1, W_2, j)$ is absolutely convergent

for $Re(s)$ large.

(ii) They are rational functions of q^{-s}

More precisely, $n=m$, $\Psi(s, W_1, W_2, f_s)$ spans a fractional ideal $\mathbb{C}[q^{-s}, q^{s}] L(s, \pi_1 \times \pi_2)$ of the ring $\mathbb{C}[q^s, q^{-s}]$.

The factor $L(s, \pi_1 \times \pi_2)$ has the form $P(q^{-s})^{-1}$, where $P \in \mathbb{C}[X]$, and $P(0)=1$.

(Similar result for $n \neq m$)

(iii) Functional equation $n=m$.

$\exists$ factor $\varepsilon(s, \pi_1 \times \pi_2, \psi)$ of the form cq^{-ns} s.t.

$\Psi(1-s, \tilde{W}_1, \tilde{W}_2, \hat{f}_s) / L(1-s, \pi_1^{\iota} \times \pi_2^{\iota})$

$= \omega_{\pi_2}(-1)^{n-1} \varepsilon(s, \pi_1 \times \pi_2, f_s)\, \Psi(s, W_1, W_2, f_s) / L(s, \pi_1 \times \pi_2)$

(Similar result for $n \neq m$).

Application:

• Eisenstein series theory

$\Rightarrow$ the analytic properties of the constant term of the Eisenstein series can be derived from our theory.

• Characterization of automorphic repns.

π irr. repn of $G_r(F_A)$ with some auxillary conditions,

π is cuspidal & automorphic iff $\forall$ automorphic repn σ of

$r=2$ it's $F_A^{\times}$ $GL_{r-2}(F_A)$, the corresponding L-fun has the appropriate analytic behavior.

References:

- The Rankin-Selberg Method: An Introduction and Survey (Daniel Bump)
- Rankin-Selberg Convolutions (Jacquet, PS, Shalika)
- Basic Rankin-Selberg (Paul Garrett).