Hamiltonian operators and free fermionic lattice models

1) Introduction

Goal: use Hamiltonian operators to prove identities about symmetric functions

Identity	Hamiltonian analogue
Cauchy	Commutation of Hamiltonians
Pieri	Action of a certain operator
Jacobi-Trudi	Wick's theorem (only for certain Hamiltonians)
Branching rule	Branching rule (adjoint to Pieri rule)

Eventually: we'll explore which lattice models have Hamiltonians, and try to prove identities about them

2) Heisenberg algebra representations

Heisenberg algebra: $\mathcal{H} = \langle B_k \mid k \in \mathbb{Z} \setminus \{0\} \rangle$

$$[B_k, B_\ell] = \begin{cases} k, & \ell = -k \\ 0, & \text{else} \end{cases}$$

Uniqueness of representations

(see e.g. Kac-Raina Bombay lectures):

Let V_1, V_2 be nonzero highest weight vectors v_1, v_2. Then there exists a unique isomorphism of $\mathcal{H}$-modules $\phi: V_1 \to V_2$ s.t. $\phi(v_1) = v_2$.

Two important repns:

1) Bosonic Fock space:

Space: algebra Λ of symmetric functions

Action: $$B_k \mapsto \begin{cases} p_{-k}, & \text{if } k \leq -1, \\ k\dfrac{\partial}{\partial p_k}, & \text{if } k \geq 1, \end{cases}$$

where p_k is the k-th power sum symmetric function

$$p_k = x_1^k + x_2^k + \dots + x_n^k$$

Highest wt vector: $\underline{1}$

By scaling the generators, and abstracting the power sum symmetric functions:

$p_k \mapsto s_k$, where $s_1, s_2, \dots$ is an algebraically independent of symmetric functions, we can write instead

$$B_k \mapsto \begin{cases} s_{-k}, & \text{if } k \leq -1, \\ k\dfrac{\partial}{\partial s_k}, & \text{if } k \geq 1, \end{cases}$$

2) Fermionic Fock space

Clifford algebra: $A = \langle \psi_i^*, \psi_i \mid i \in \mathbb{Z} - \frac{1}{2} \rangle$

Relations: $\psi_i \psi_j + \psi_j \psi_i = 0$

$$\psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0$$

$$\psi_i \psi_j^* + \psi_j^* \psi_i = \delta_{i,j}$$

$$W := \bigoplus_{i \in \mathbb{Z}} \mathbb{C}\psi_i^* \oplus \bigoplus_{i \in \mathbb{Z}} \mathbb{C}\psi_i$$

$$W_{ann} := \bigoplus_{i<0} \mathbb{C}\psi_i^* \oplus \bigoplus_{i>0} \mathbb{C}\psi_i$$

$$W_{cr} := \bigoplus_{i>0} \mathbb{C}\psi_i^* \oplus \bigoplus_{i<0} \mathbb{C}\psi_i$$

$$\mathcal{F} := A / A W_{ann} \qquad \mathcal{F}^* := A W_{cr} \backslash A$$

Cylic A modules, with generators

$$|0\rangle \quad \text{and} \quad \langle 0|$$

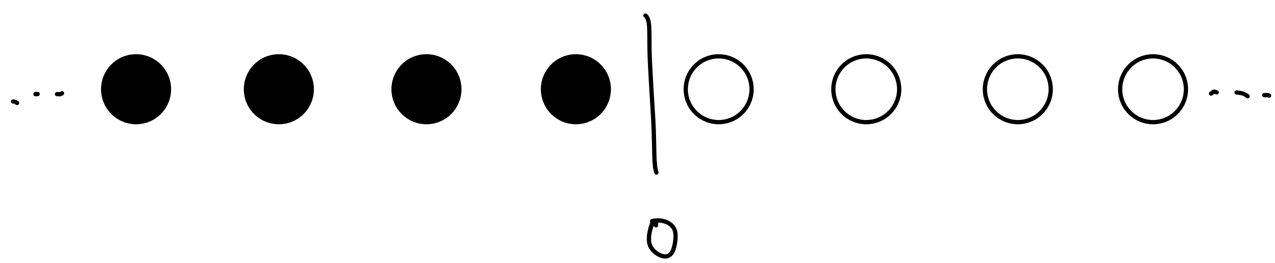
Symm. bilinear pairing:

$$\langle 0| a \otimes_A b |0\rangle$$

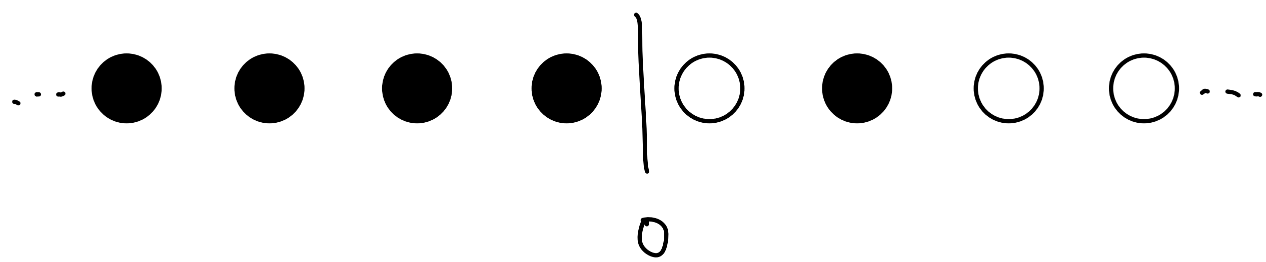
Dirac sea: each basis vector of $\mathcal{F}$ is a state of particles

e.g.

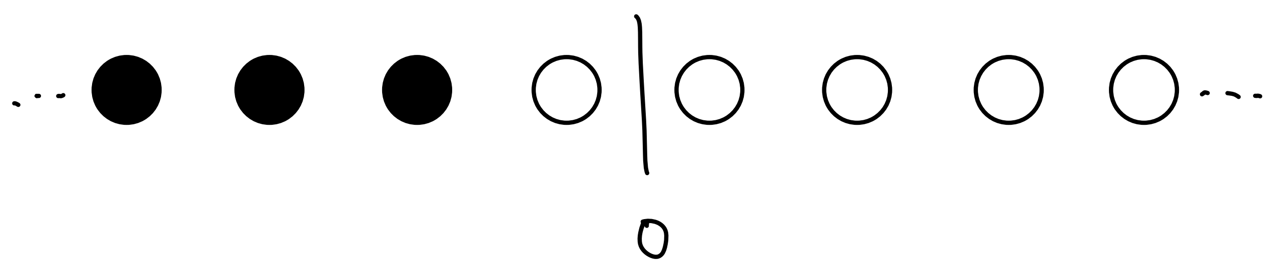
$|0\rangle$:



$\psi^*_{3/2}|0\rangle$:



$\psi_{-1/2}|0\rangle$:



Let $\mathfrak{gl}(\infty) := \left\{ \sum_{i,j} a_{i,j} \underbrace{:\psi_i^* \psi_j:}_{\text{"normal ordering"}} \;\middle|\; \exists N \text{ s.t. } a_{i,j} = 0 \text{ if } |i-j| > N \right\}$

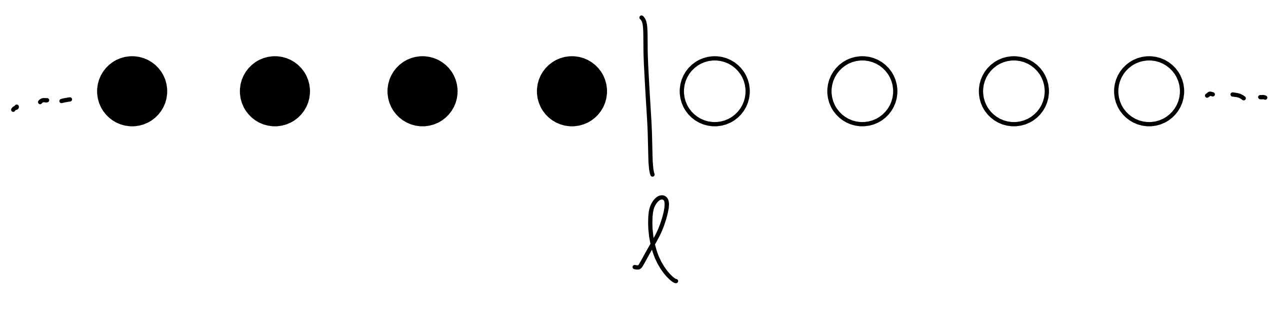
$\mathcal{F}$ decomposes into $\mathfrak{gl}(\infty)$-irreps

$$\mathcal{F} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{F}_\ell$$

where $\mathcal{F}_\ell$ has highest weight vector

$$|\ell\rangle = \begin{cases} \psi_{\ell+\frac{1}{2}} \cdots \psi_{-\frac{1}{2}} |0\rangle & , \text{ if } \ell < 0 \\ |0\rangle & , \text{ if } \ell = 0 \\ \psi^*_{\ell-\frac{1}{2}} \cdots \psi^*_{1/2} |0\rangle & , \text{ if } \ell > 0 \end{cases}$$

$|\ell\rangle$:



If λ is a strict partition with ℓ parts:

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell),$$

then

$$|\lambda\rangle := \psi^*_{\lambda_1} \psi^*_{\lambda_2} \cdots \psi^*_{\lambda_\ell} |0\rangle \in \mathcal{F}_\ell$$

$$\langle\lambda| := \langle 0| \psi_{\lambda_\ell} \psi_{\lambda_{\ell-1}} \cdots \psi_{\lambda_1} \in \mathcal{F}_\ell^*$$

Note that $\langle\mu|\lambda\rangle = \begin{cases} 1, & \lambda = \mu \\ 0, & \text{else} \end{cases}$

Current operators:

$$J_n = \sum_{i \in \mathbb{Z} - \frac{1}{2}} :\psi^*_{i-n} \psi_i: \qquad \text{(move one particle } n \text{ spaces left)}$$

The current operators form a Heisenberg algebra:

$$[J_m, J_n] = m\, \delta_{m,-n}$$

$$H[s] = \sum_{n=1}^{\infty} s_n J_n \,, \quad e^{H[s]} = \sum_{k=0}^{\infty} \frac{1}{k!} H[s]^k$$

Boson-Fermion correspondence: Let $B_k = s_k J_k$

The map $a|0\rangle \mapsto \langle \ell | e^{H[s]} a|0\rangle$

is an isomorphism of $\mathcal{H}$-modules, In other words,

$$\langle \ell | e^{H[s]} B_k a|0\rangle = B_k \langle \ell | e^{H[s]} a|0\rangle$$

and $\langle \mu+\rho | e^{H[p]} | \lambda+\rho\rangle = s_{\lambda/\mu}$

Wick's Theorem:

$$\langle \ell | \psi_{i_1} \cdots \psi_{i_n} e^{H} \psi^*_{j_1} \cdots \psi^*_{j_n} | \ell \rangle$$
$$= \det_{1\le a,b\le n} \langle \ell | \psi_{i_a} e^{H} \psi^*_{j_b} | \ell\rangle$$

$$\langle \mu | e^{H} | \lambda \rangle = \det_{1\le a,b\le \ell} \langle 0 | \psi_{\mu_a} e^{H} \psi^*_{\lambda_b} | 0 \rangle$$

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3) Identities from Hamiltonians

Now, let

$$H := H_+ := H_+[s] = \sum_{k\geq 1} s_k J_k$$

where $s_k = \sum_{j=1}^{n} s_k^{(j)}$

If $s_k^{(j)} = x_j^k$, then $s_k = p_k$

Let $H_- := H_-[s] = \sum_{k\leq -1} s_k J_k$, where $s_k = \sum_{j=1}^{n} s_k^{(j)}$

Cauchy identity

Let $F_{\lambda/\mu} = \langle \lambda | e^{H_-} | \mu \rangle$

$G_{\lambda/\mu} = \langle \mu | e^{H_+} | \lambda \rangle$

Then,

$$\sum_{\nu} G_{\nu/\lambda} F_{\nu/\mu} = \sum_{\nu} \langle \lambda | e^{H_+} | \nu \rangle \langle \nu | e^{H_-} | \mu \rangle$$

$$= \langle \lambda | e^{H_+} e^{H_-} | \mu \rangle$$

$$= e^{[H_+, H_-]} \langle \lambda | e^{H_-} e^{H_+} | \mu \rangle$$

$$= e^{[H_+, H_-]} \sum_{\nu} \langle \lambda | e^{H_-} | \nu \rangle \langle \nu | e^{H_+} | \mu \rangle$$

$$= e^{[H_+, H_-]} \sum_{\nu} F_{\lambda/\nu} G_{\mu/\nu}$$

Now, let $\eta_k := \langle 0 | e^{H} | (k) \rangle$

$$V_k = \sum_{\lambda \vdash k} z_\lambda^{-1} B_{\lambda_1} B_{\lambda_2} \cdots B_{\lambda_\ell},$$

where $z_\lambda = 1^{m_1(\lambda)} m_1(\lambda)! \, 2^{m_2(\lambda)} m_2(\lambda)! \cdots$

$$m_i(\lambda) = |\{ j \mid \lambda_j = i \}|$$

Pieri rule:

$$\eta_k G_{\lambda/\rho} = \sum_{\mu} \langle \mu | U_k | \lambda \rangle G_{\mu/\rho}$$

Jacobi-Trudi identity:

$$G_{\lambda/\mu} = \langle \mu | e^{H} | \lambda \rangle$$

$$= \det_{1 \le a,b \le n} \langle 0 | \psi_{\mu_a} e^{H} \psi^*_{\lambda_b} | 0 \rangle$$

$$= \det_{1 \le a,b \le n} \eta_{\lambda_b - \mu_a}$$

Branching rule:

Let $\phi^{(j)} = \sum_{k \ge 0} s^{(j)}_k J_k$ so that $H_+ = \sum_{j=1}^{n} \phi^{(j)}$

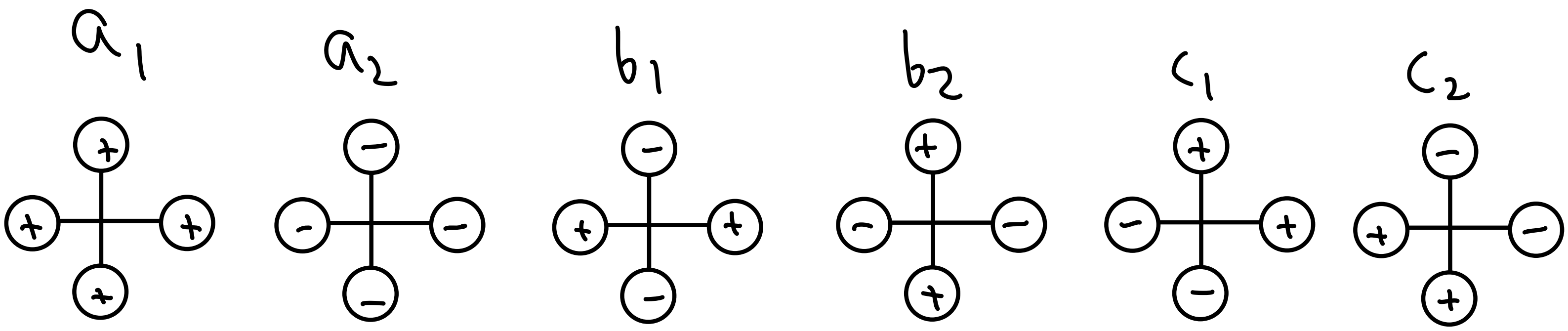
$$G_{\lambda/\mu} = \langle \mu | e^{H} | \lambda \rangle$$

$$= \langle \mu | e^{\phi^{(n)}} \cdots e^{\phi^{(1)}} | \lambda \rangle$$

$$= \sum_{\nu} \langle \mu | e^{\phi^{(n)}} | \nu \rangle \langle \nu | e^{\phi^{(n-1)}} \cdots e^{\phi^{(1)}} | \lambda \rangle$$

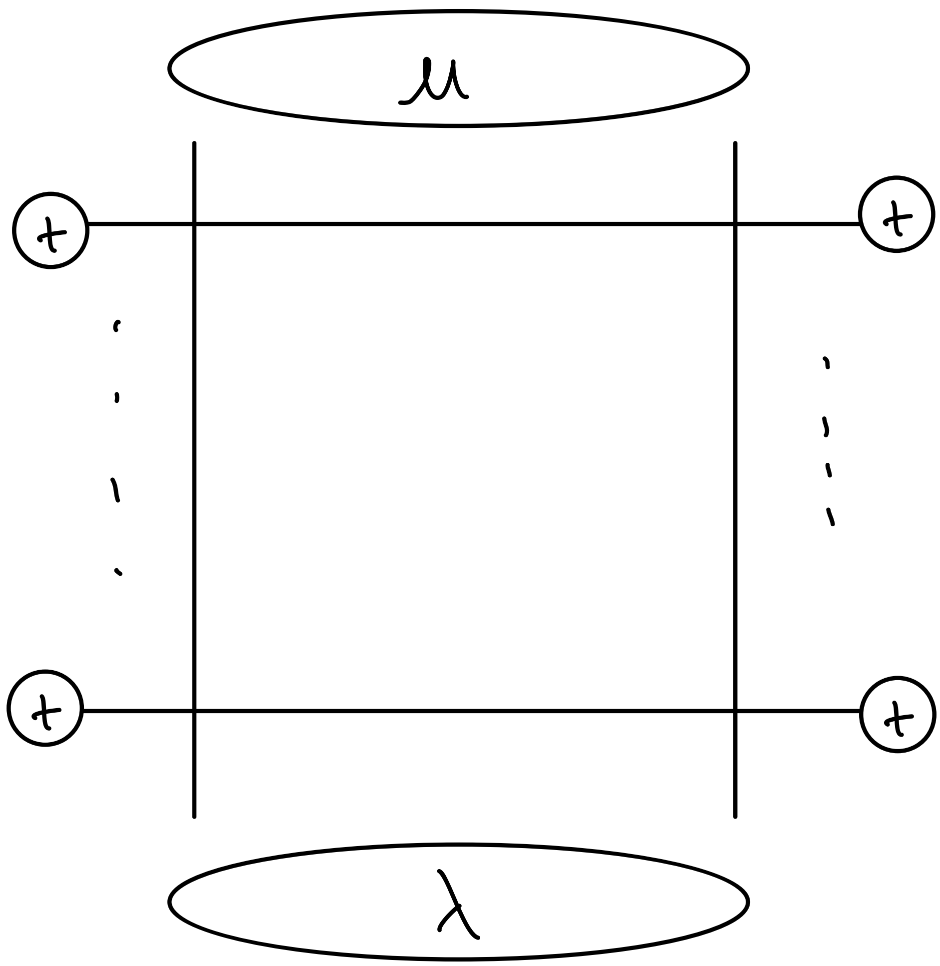
4) Lattice models and Hamiltonian operators

Let $\mathfrak{S}_{\lambda/\mu}$ be a lattice model with n rows, m columns $(m \geq \lambda_1, \mu_1)$, and admissible vertices:



and boundary conditions:

$(\ell(\lambda) = \ell(\mu) = \ell)$



When do there exist parameters $s_k^{(j)}$ such that $Z_{\lambda/\mu} := Z(\mathfrak{S}_{\lambda/\mu})$ corresponds to a Hamiltonian operator?

Answer: when the Boltzmann weights are <u>free fermionic</u>: $a_1a_2 + b_1b_2 = c_1c_2$

Let $$s_k^{(j)} = \frac{1}{k}\left(\left(\frac{b_2^{(j)}}{a_1^{(j)}}\right)^k + (-1)^{k-1}\left(\frac{a_2^{(j)}}{b_1^{(j)}}\right)^k\right)$$

Then if $a_1a_2 + b_1b_2 = c_1c_2$, we have:

$$Z_{\lambda/\mu} = \left(\prod_{i=1}^{n} a_1^{(i)\,m-\ell}\; b_1^{(i)\,\ell}\right) \cdot \langle \mu | e^{H} | \lambda \rangle$$

So free fermionic lattice models correspond to Hamiltonians of free fermions...

Sketch of Proof: First treat 1-particle systems: $\ell(\lambda) = \ell(\mu) = 1$. We can prove that the result holds for these partitions precisely when $s_j^{(k)}$ is given by the expression above.

For multiple particles, do the $n=1$ case first, and reduce to the 1-particle case via Wick's theorem. The determinant coming from the Hamiltonian equals the one-row partition function precisely when the weights are free fermionic!

Takeaways:

1) Let $x_j := \dfrac{b_2^{(j)}}{a_1^{(j)}}$, $y_j := -\dfrac{a_2^{(j)}}{b_1^{(j)}}$

Then we have

$$s_k^{(j)} = \frac{1}{k}\left(x_j^k - y_j^k\right) = \frac{1}{k} p_k(x|y),$$

the superalgebra analogue of the power sum symmetric function

$$\left\{\begin{array}{l}\text{Free fermionic} \\ \text{Boltzmann} \\ \text{weights}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{Hamiltonians} \\ \text{of the} \\ \text{above form}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{Supersymmetric} \\ \text{Schur functions}\end{array}\right\}$$

(Brubaker–Schultz showed $\leftarrow$)

2) We obtain Cauchy, Pieri, Jacobi–Trudi identities for supersymmetric Schur functions.

3) Edrei–Thoma theorem: A homomorphism $\Lambda \to \mathbb{R}$ is Schur-positive if and only if

$$p_1 \mapsto \gamma + \sum_j (x_j + y_j)$$

$$p_k \mapsto \sum_j \left(x_j^k + (-1)^{k-1} y_j^k\right), \quad k \geq 2,$$

where $\gamma, x_j, y_j \geq 0$

Corollary: Let $a_1^{(j)}, a_2^{(j)}, b_1^{(j)}, b_2^{(j)}, c_1^{(j)}, c_2^{(j)} \in \mathbb{R}$, and let $a_1^{(j)} a_2^{(j)} + b_1^{(j)} b_2^{(j)} = c_1^{(j)} c_2^{(j)}$.

Then $Z_{\lambda/\mu} \geq 0$ for all λ, μ

$$\Updownarrow$$

$$\frac{b_2^{(j)}}{a_1^{(j)}} \geq 0, \quad \frac{a_2^{(j)}}{b_1^{(j)}} \geq 0 \quad \text{for all } 1 \leq j \leq n$$

Question: can this be generalized to the non-free fermionic case? Is there a relationship between Δ and γ?

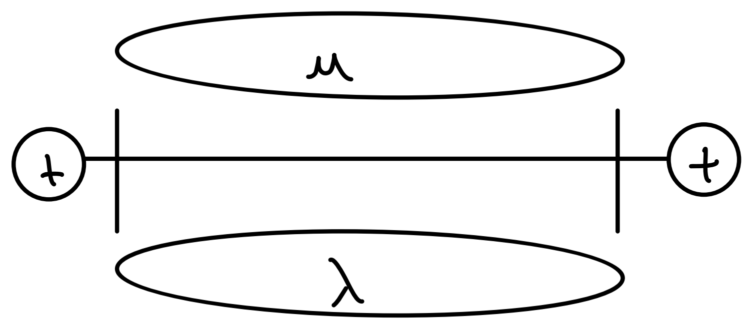
Boundary conditions

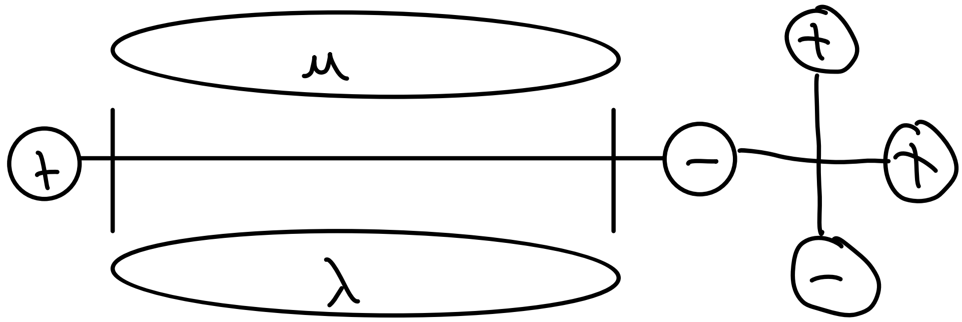
Two ways to deal with them

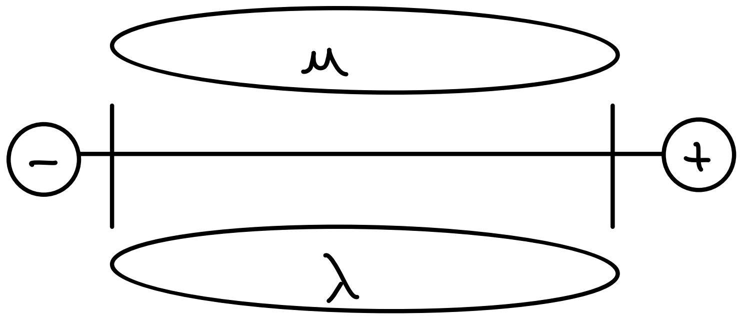
1) Use creation/deletion operators

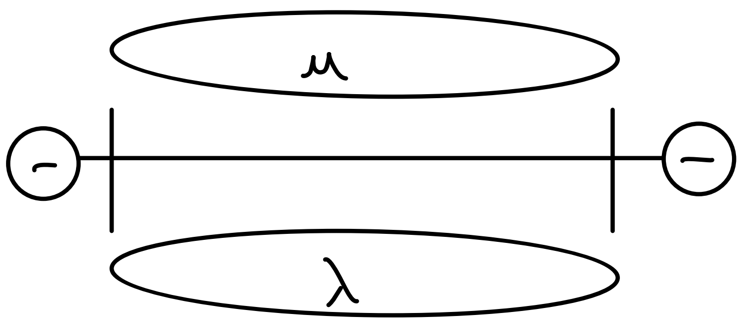
Let:

$A_{\lambda/\mu} :=$



$B_{\lambda/\mu} :=$ 

$C_{\lambda/\mu} :=$ 

$D_{\lambda/\mu} :=$ 

Then if $A_{\lambda/\mu} = \langle \mu | e^{\phi} | \lambda \rangle$, then

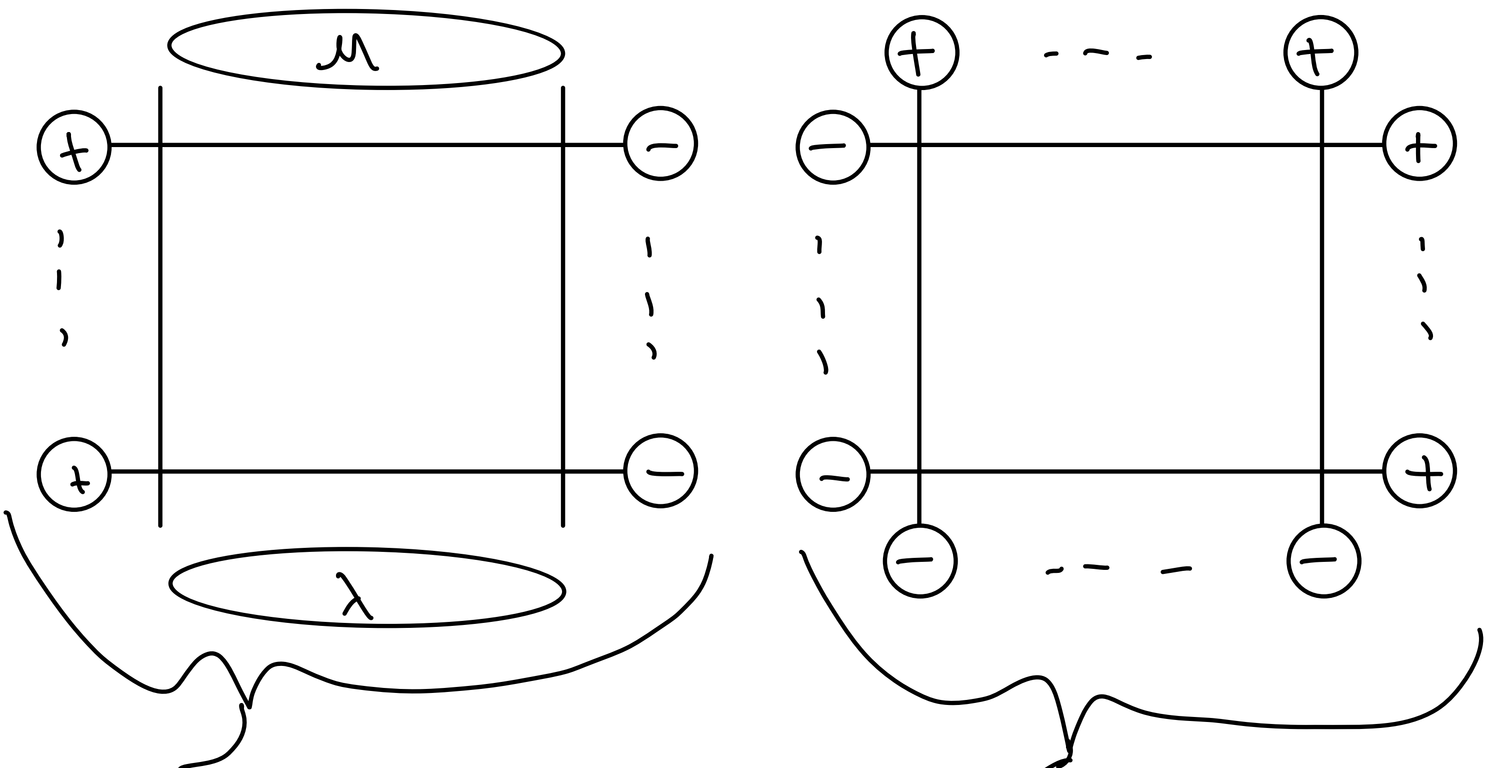
$$B_{\lambda/\mu} = \frac{1}{c_1}\langle \mu | e^{\phi} \psi^*_{m+1/2} | \lambda \rangle$$

$$C_{\lambda/\mu} = \frac{1}{c_2}\langle \mu | \psi_{-1/2} e^{\phi} | \lambda \rangle$$

$$D_{\lambda/\mu} = \frac{1}{c_1 c_2} \langle \mu | \psi_{-1/2} \, e^{\phi} \, \psi^*_{m+1/2} | \lambda \rangle$$

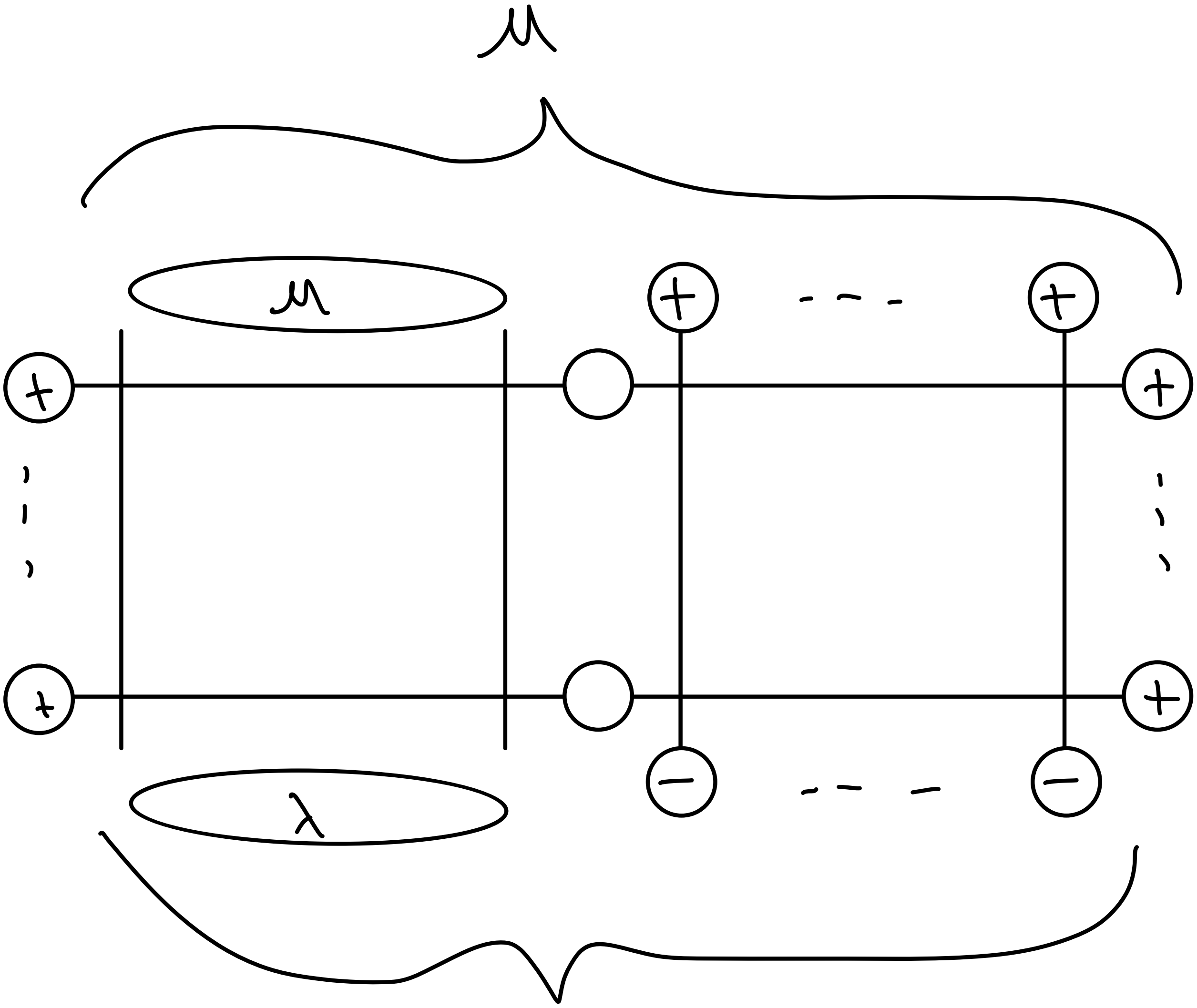
"ghost vertices"

2) For "solid" boundary conditions, add on an extra block



Z

$$\prod_{i=1}^{n} c_2^{(j)} \prod_{i<j} \left(a_1^{(i)} a_2^{(j)} + b_1^{(j)} b_2^{(i)} \right)$$



$$\lambda' = (m+n,\ m+n-1,\ \cdots,\ m+1,\ \lambda_1,\ \cdots,\ \lambda_\ell)$$

$$Z = \frac{\left(\prod_{i=1}^{n} a_1^{(i)\,m-\ell}\ b_1^{(i)\,\ell+n}\right) \langle \mu | e^{H} | \lambda' \rangle}{\prod_{i=1}^{n} c_2^{(j)} \prod_{i<j} \left(a_1^{(i)} a_2^{(j)} + b_1^{(j)} b_2^{(i)}\right)}$$

5) q-Fock space and lattice models with charge

(Brubaker - Buciumas - Bump - Gustafsson)

KMS Fock space associated to $U_q(\widehat{\mathfrak{sl}}_n)$

$$\mathcal{F}_0 = \langle u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots \mid i_m = m \text{ for } m \ll 0 \}$$

$$u_\ell \wedge u_m = \begin{cases} -u_m \wedge u_\ell, & \text{if } \ell \equiv m \mod n \\ g(\ell-m)\, u_m \wedge u_\ell + \cdots, & \text{otherwise} \end{cases}$$

$(\ell < m)$

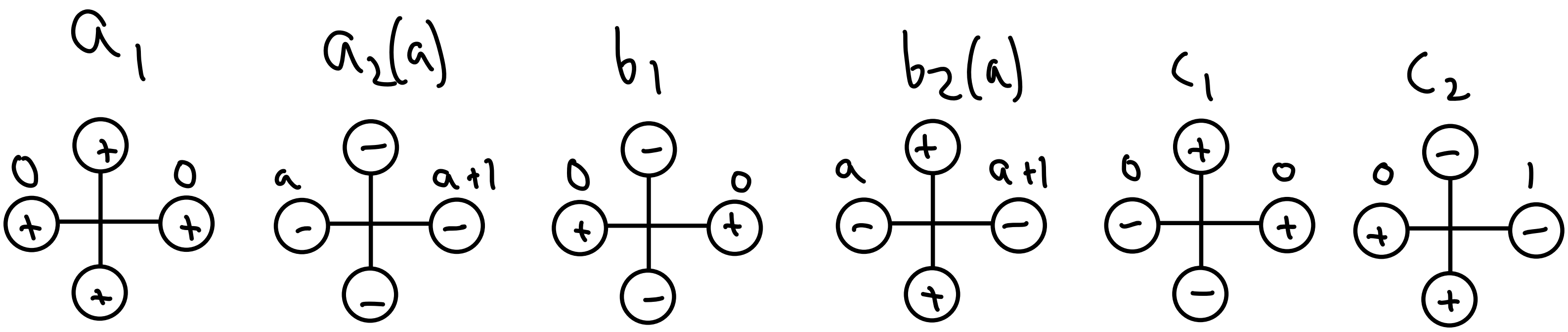
$$g(0) = -v \quad (v = -q^2)$$

$$g(a)\, g(-a) = v, \text{ if } a \not\equiv 0 \mod n$$

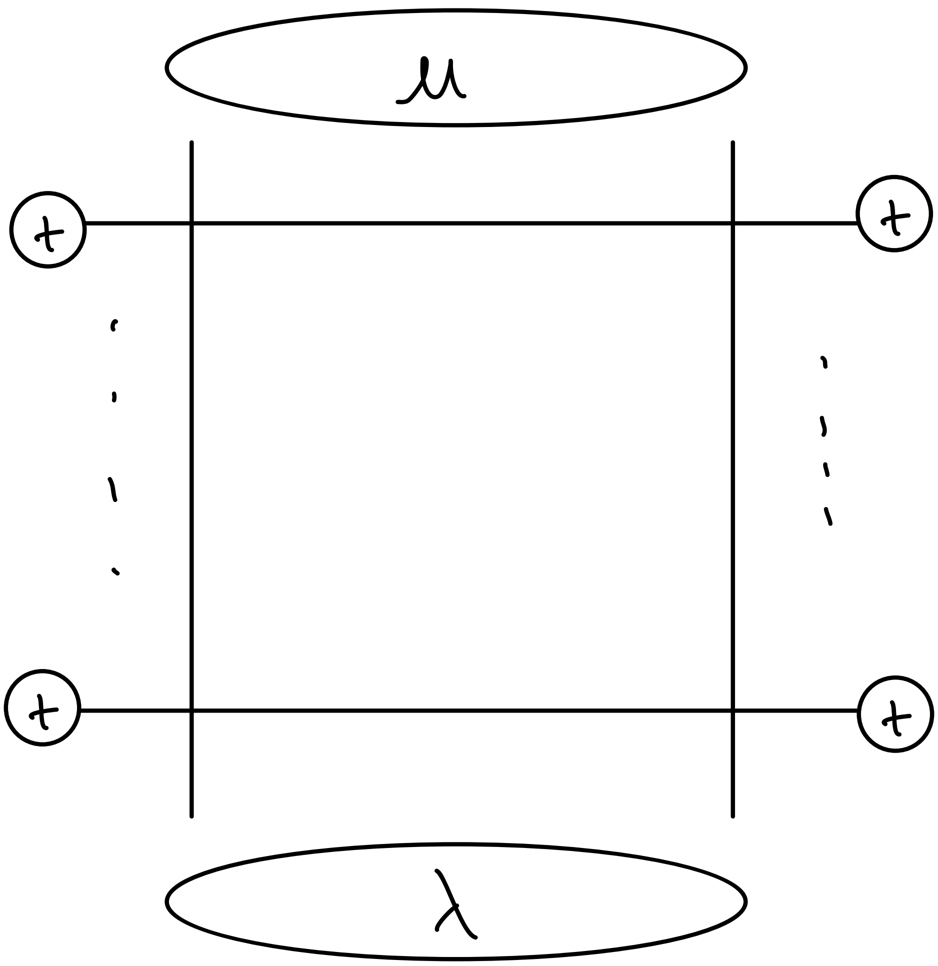
$$J_k (u_{i_m} \wedge u_{i_{m-1}} \wedge \cdots)$$

$$= u_{i_m - nk} \wedge \cdots + u_{i_m} \wedge u_{i_{m-1} - nk} \wedge \cdots + \cdots$$

Consider a 6-vertex model with charge:



a mod n



When does this lattice model match a q-Fock space Hamiltonian?

(Note: we will let v vary. However, the q-Fock space may become degenerate if $v=0$ or a root of 1.)

Answer: when the Boltzmann weights satisfy the following conditions:

- $a_1 a_2(0) + b_1 b_2(0) - c_1 c_2 = 0$
- $\dfrac{a_2(0)}{b_2(0)} \neq 0$ or a root of 1
- For any $1 \leq a \leq n-1$, $\dfrac{a_2(0)}{b_2(0)} = -\dfrac{a_2(a)\, a_2(-a)}{b_2(a)\, b_2(-a)}$

We'll call this the "generalized free fermion condition".

Set $s_k^j = \frac{1}{k}\left(\prod_a \left(\frac{b_2^{(j)}(a)}{a_1^{(j)}}\right)^k + (-1)^{k-1} \prod_a \left(\frac{a_2^{(j)}(a)}{b_1^{(j)}}\right)^k\right)$

Then,

$$Z_{\lambda/\mu} = \underline{\qquad\qquad} \langle \mu | e^H | \lambda \rangle$$

and we obtain the supersymmetric LLT polynomials:

$$Z_{\lambda/\mu} = \underset{\sim}{G}_{\lambda/\mu}(x|y)$$

where $x_j = \prod_a \frac{b_2^{(j)}(a)}{a_1^{(j)}}$, $y_j = \prod_a \frac{a_2^{(j)}(a)}{b_1^{(j)}}$

Note that we <u>cannot</u> have $y = 0$ since this would entail setting $v = 0$, So we cannot obtain the regular LLT polynomial

$$\underline{\mathcal{G}}_{\lambda/\mu}(x) := \underline{\mathcal{G}}_{\lambda/\mu}(x|0)$$

in this way.

Identities: can do Cauchy, Pieri, but not Jacobi-Trudi.

Boundary conditions: can do method 1, but not method 2.