1) Semisimple Monoids
2) ~~Class.~~ Repns of Reductive Groups
3) Polyhedral Root Systems
4) Renner's Classification, and Examples
5) Sketch of Proof

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1) Semisimple Monoids
(see $\boxed{1\text{-}2}$)

2) Repns of Reductive Groups
(see $\boxed{2\text{-}3}$)

3) Polyhedral Root Systems

From now on, G_0: semisimple gp., $G = G_0 \times K^*$
T_0 : max'l torus of G_0 (resp T) (resp G)

M: semisimple monoid w/ D-monoid (closure of max'l torus) Z.

Idea: to classify M, use root system of G_0, and also $X(Z)$.

Def: A polyhedral root system of semisimple rank n is a triple (X, Φ, C) where

1) X free abelian gp. of rank $n+1$
2) $\Phi \subseteq X$ spans a subgp. $\langle \Phi \rangle$ of rank n
3) $C \subseteq X$ is $X \cap \sigma$ w/ σ: rational poly. cone of $X \otimes \mathbb{Q}$ of dim $n+1$
4) (X_0, Φ) is root system, $X_0 := \{x \in X \mid mx \in \langle \Phi \rangle, m \in \mathbb{Z}\}$
5) The action of W on Φ extends to X and fixes C.

1) ~~Def~~ semisimple monoids

Def: A monoid is a set w/ an assoc. binary operator and an identity.

Def: ~~An algebra~~ Let K be an algebraically-closed field. An algebraic monoid is an monoid that is also an ~~irred.~~ affine variety / K.

Def: An algebraic monoid M is reductive ~~semisimple~~ if it is irred. and has a connected reductive group of units G.
M is semisimple if further M is normal as a variety, has a 0, and has dimension 1 center.

Think: $M = \mathrm{Mat}_n(K)$
$G = GL_n(K)$

M: semisimple monoid, G: gp of units, T: max'l torus, B: Borel subgp

Define: ~~T~~ ~~M max'l~~

Facts

$M = \langle G, E(\overline{T}) \rangle$ $\{E$: idempotents of ~~max'l torus T of G~~ $\overline{T}$

Let $R := \overline{N_G(T)}/T$. Then $M = \bigsqcup_{r \in R} BrB$

Bruhat-Renner decomp

Every Borel subgp of G is the left centralizer of a maximal chain of idempotents.

2) Representations of Reductive Groups

G_0: semisimple gp.

G_0 has a root system $\Phi \subseteq X(T)$: characters of torus

W action $W \circlearrowright X(T)$

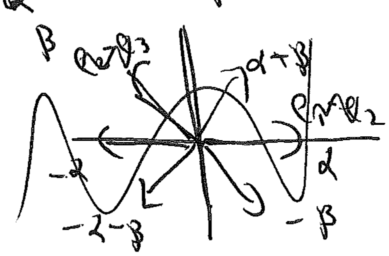
This is most of classification of semisimple groups

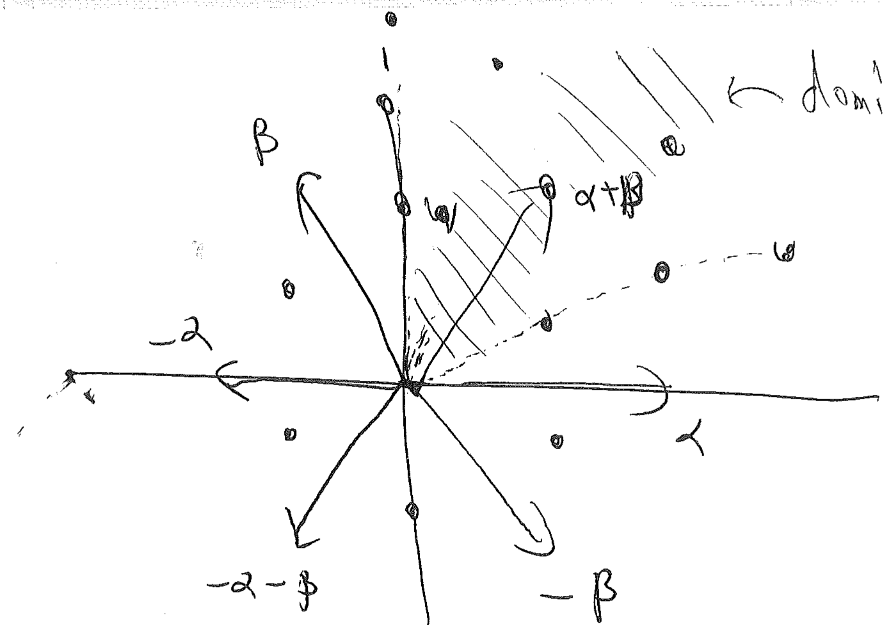
Dominant weights: $X^+(T) := \{\lambda \in X(T) \mid \langle \lambda, \alpha \rangle \geq 0 \ \forall \alpha \in \Delta\}$ simple roots

Thm (Chevalley): Irreps of $G \Leftrightarrow$ dominant wts. of T.

Ex: $G = SL_3$ $\Phi = \{e_i - e_j \mid i \neq j \ 1 \leq i,j \leq 3\}$

$\Delta = \{\underbrace{e_1 - e_2}_{\alpha}, \underbrace{e_2 - e_3}_{\beta}\}$





3) Polyhedral root systems

Construction:

~~Idea~~. G_0: semisimple gp. $G = G_0 \times K^*$

Repn $\{ \rho^\lambda_n : G_0 \times K^* \to GL(V)$

$$\rho^\lambda_n(g, t) := \rho^\lambda(g) \begin{bmatrix} t^n & & \\ & \ddots & \\ & & t^n \end{bmatrix}$$

highest wt repn of assoc. to λ.

Set $M := \overline{\rho^\lambda_n(G_0 \times K^*)}$

Renner's idea: look at $X(\overline{T})$.

Def: The polyhedral root system of a semisimple monoid w/ unit gp. G is $(X(T), \Phi(G_0), X(Z))$

Now let's construct some monoids!

Idea: take a repn of ρ of G, then take closure $M := \overline{\rho(G)}$

Let $\Phi(\rho) \subseteq X(T)$ be the set of wts of $\rho|_T$

Prop (Renner): $X(Z) = \langle \Phi(\rho) \rangle$

Def: $\rho_n^\lambda : G \to GL(V)$

$$\rho_n^\lambda(g,t) = \rho(g)\begin{bmatrix} t^n & & \\ & \ddots & \\ & & t^n \end{bmatrix}$$

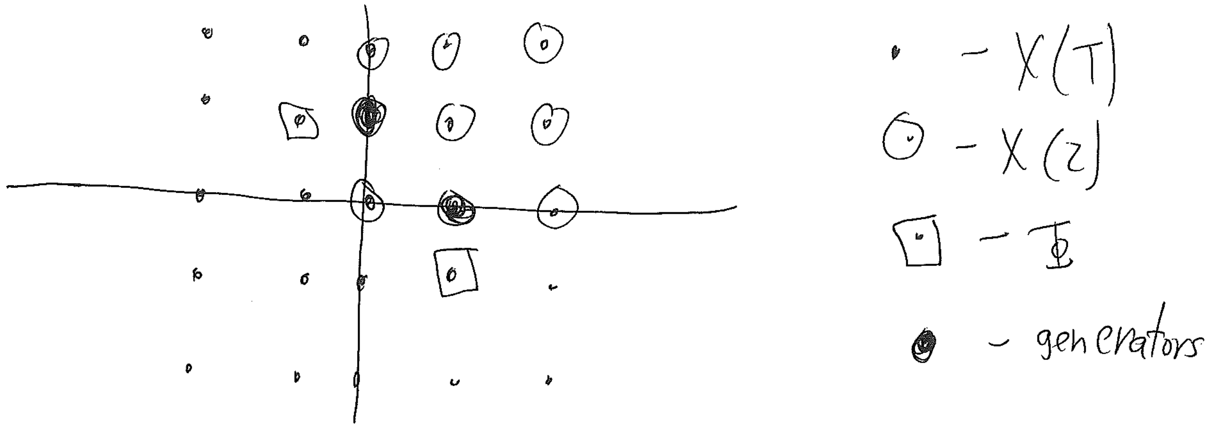
Prop ?: $\Phi(\rho_n^\lambda) \subseteq$ rational convex hull of $W \cdot (\lambda, n)$

4) Renner's Classification, and Examples

Ex: 1) Let $M = Mat_2(k)$. Then, $G = GL_2(k)$ $\quad T = \{\begin{pmatrix} a & \\ & b\end{pmatrix} \mid ab \neq 0\}$ $\quad X(T) = \mathbb{Z} \oplus \mathbb{Z}$

$Z = \{\begin{pmatrix} a & \\ & b\end{pmatrix}\}$ $\quad X(Z) = \mathbb{N} \oplus \mathbb{N}$

$\Phi = \{\pm(1,-1)\}$

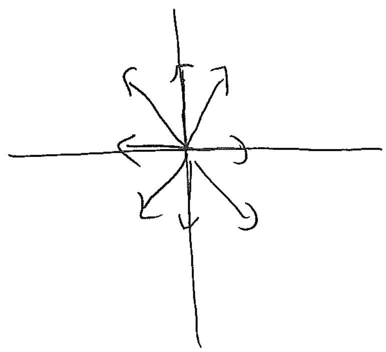


2) (see $\boxed{6}$)

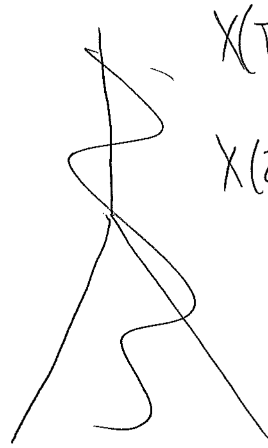
2) $G_0 = B_2(k)$ ← adjoint simple gp of type B_2

$\rho = \mathrm{Ad} : G_0 \to GL(\mathfrak{g})$

$M :=$ normalization of $\overline{\rho(G_0) \cdot k^*}$

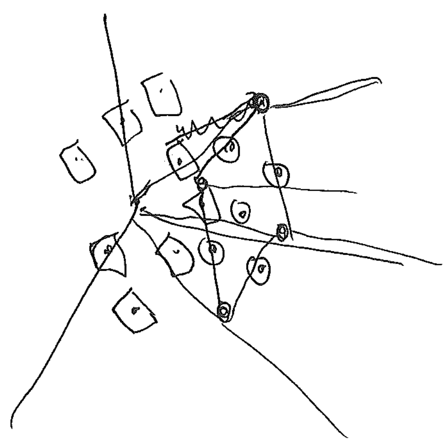


B_2



$X(T) \cong \mathbb{Z}^3$

$X(Z) = \langle (\alpha, 1) \rangle_{\alpha \in \Phi}$



Classification Theorem (Renner): For every poly root sys (X, Φ, C), $\exists !$ semisimple monoid M s.t. $(X(T), \Phi, X(Z)) \cong (X, \Phi, C)$.

5) Sketch of Proof

Existence: (see $\boxed{7}$)

Uniqueness: (see $\boxed{8}$)

Sketch of proof:

a) Let (X, Φ, C) be a polyhedral root system.
Let $G = G_0 \times K^*$, where G has root system (X, Φ) w.r.t. T max'l torus.
Gordon's Lemma: C gen'd by finite W-stable subset

$$S = \{(\lambda_i, n_i)\}_{i=1}^{s} \subseteq C \subseteq X(T) = X(T_0) \oplus \mathbb{Z}$$

Take

$$S^+ = \{(\lambda, n) \in S \mid \lambda \text{ dom.}\}$$

Define $\rho := \bigoplus_{(\lambda, n) \in S^+} \rho_n^\lambda$

Then $\Phi(\rho) \subseteq C$, but also generates C, so

$$X(\overline{\rho(T)}) = C$$

~~Let M be the normalization of $\overline{\rho(G)}$. Then~~

~~And~~ ~~Then~~ $\overline{\rho(G)}$ has poly. root system (X, Φ, C), and taking M = normalization of $\overline{\rho(G)}$ gives a semisimple monoid w/ same poly root sys.

b) Thm (Extension Principle): Let E be reductive, normal, and suppose $G \to E'$, $\beta : Z \to E'$ are morphisms of alg. monoids w/ $\alpha|_T = \beta|_T$. Then $\exists!\ \varphi : E \to E'$ s.t. $\varphi|_G = \alpha$, $\varphi|_Z = \beta$.
(Proof: ~~Results~~ Look at "big cell" of E).

Suppose ~~E~~ $M, M' \longleftrightarrow (X, \Phi, C)$.

Comm. diag.

$$
\begin{tikzcd}
G \arrow[rr, hook, "\alpha"] \arrow[dd] & & M' \\
& T \arrow[ul] \arrow[r, hook] \arrow[d] & Z' \arrow[u, hook] \\
M & Z \arrow[l, hook'] \arrow[ur, "\beta"'] &
\end{tikzcd}
$$

$\beta \longleftarrow$ normalization, since $X(Z) = X(Z')$

Have morphisms $G \to M'$
$Z \to M'$, so we obtain

$$
\begin{tikzcd}
G \arrow[d] \arrow[dr, "\alpha"] & \\
M \arrow[r, dashed, "\exists!\varphi"] & M' \\
Z \arrow[u, hook] \arrow[ur, "\beta"'] &
\end{tikzcd}
$$

By geometric facts (Zariski's Main Theorem), φ is an isom.