Iwahori-Hecke Algebras in Multiple Contexts

1) Hecke algebras for a reductive group
2) Presentation of spherical/finite/affine Hecke algebras
3) Quantum Schur-Weyl duality

1) Reductive Groups

Definitions

G: reductive gp/F: nonarch. local field

$\mathcal{O}$: ring of ints. of F

$\mathfrak{p}$: max'l ideal of $\mathcal{O}$

B = Borel subgp.

K° = max'l compact subgp

J = Iwahori subgp.

Favorite example

$G = GL_n(\mathbb{Q}_p)$

$\mathcal{O} = \mathbb{Z}_p$

$\mathfrak{p} = \langle p \rangle$

$$B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$$

$$K^\circ = \begin{pmatrix} \mathcal{O} & \cdots & \mathcal{O} \\ \vdots & & \vdots \\ \mathcal{O} & \cdots & \mathcal{O} \end{pmatrix}$$

$$J = \begin{pmatrix} \mathcal{O} & & \mathcal{O} \\ & \ddots & \\ \mathfrak{p} & & \mathcal{O} \end{pmatrix}$$

Let K be a compact open subgp. of G. The Hecke alg. of G relative to K is the set of smooth cmptly supp. K-biinvariant funs on G:

$$\mathcal{H}_K := \{\phi : G \to \mathbb{C}, \text{smooth cpt. supp.} \mid \phi(kgk') = \phi(g)\ \forall k, k' \in K,\ g \in G\},$$

w/ mult. defined as convolution.

Remarks

1) Reductive groups are <u>hard</u>
2) Hecke alg. are relatively simple: often finite(-ish) dim'l.
3) Borel–Matsumoto: $\exists$ corresp. btwn irreps of $\mathcal{H}_K$ and "admissible" irreps of G w/ a <u>K-fixed vector</u> v ($k \cdot v = v\ \forall k \in K$).
4) So Hecke algebras are a tool to understand repns of red. gps.

But what do Hecke algebras look like?

2) Presentations (Iwahori)

For this section, $G = GL_n$, (but can be done for any Cartan type).

$$\mathcal{H}_{K^\circ} = X_*(T) \cong \mathbb{Z}^n \quad \text{(spherical Hecke alg.)}$$

cochar lattice

$$\mathcal{H}_B = \left\langle T_i,\ i=1,\dots,n-1 \;\middle|\; \begin{array}{l} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \\ T_i T_j = T_j T_i,\ i \neq j \pm 1 \\ T_i^2 = (q-1) T_i + q \end{array} \right\rangle$$

(finite Hecke alg.)

$$\mathcal{H}_J = \left\langle T_i,\ i=0,\dots,n-1 \;\middle|\; \begin{array}{l} \text{same reln's as for } \mathcal{H}_B, \\ \text{but indices taken mod } n \end{array} \right\rangle$$

(affine Hecke alg.)

Remarks

1) <u>Not</u> guaranteed a simple presentation of $\mathcal{H}_K$ for other subgps. K, <u>but</u>

2) $\mathcal{H}_{K^\circ}$ is commutative!

3) $\mathcal{H}_B$ is a deformation of the gp. alg of S_n:

If $q \mapsto 1$

$$\mathcal{H}_B \mapsto \mathbb{C}[S_n].$$

So repn theory of finite Hecke algebras relate to repn. theory of S_n.

4) Exact sequences:

$$\begin{tikzcd} 1 \arrow[r] & ? K^\circ \arrow[r] & J \arrow[r] \arrow[d, squiggly] & B(\mathbb{F}_q) \arrow[r] & 1 \\ 0 \arrow[r] & \mathcal{H}_{K^\circ} \arrow[r] & \mathcal{H}_J \arrow[r] & \mathcal{H}_B \arrow[r] & 0 \end{tikzcd}$$

So to understand $\mathcal{H}_J$, want to understand $\mathcal{H}_{K^\circ}, \mathcal{H}_B$.

3) <u>Quantum Schur-Weyl Duality</u>

First, classical S-W duality:

Let $V = \mathbb{C}^n$ be std. repn of $G = GL_n(\mathbb{C})$.

Now, take $V^{\otimes k}$, for $k \le n$, and let G act diagonally:

$$g \cdot (v_1 \otimes \cdots \otimes v_k) := g \cdot v_1 \otimes \cdots \otimes g \cdot v_k.$$

Let S_k act on $V^{\otimes k}$ by permuting the factors:

$$(v_1 \otimes \cdots \otimes v_k) \cdot \sigma = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$$

These actions commute, and in fact are <u>mutual centralizers</u>.

Schur-Weyl Duality: As a (GL_n, S_k)-bimod, $V^{\otimes k}$ decomposes as

$$V^{\otimes k} = \bigoplus_{\lambda \vdash k} L^\lambda \otimes S^\lambda,$$

where the L^λ are (distinct) highest wt. modules, and the S^λ are (distinct) Specht mods.

Now, let V be the std. repn. of the quantum gp. $U := U_q(\mathfrak{gl}_n)$, $q \neq$ root of unity, and let U act on $V^{\otimes k}$ via the <u>coproduct</u> map.

Since U not cocomm., we can't just permute the factors. Instead, we use the Yang-Baxter eqn. to define isomorphisms:

$$R_i : V_1 \otimes \cdots \otimes V_i \otimes V_{i+1} \otimes \cdots \otimes V_k \cong V_1 \otimes \cdots \otimes V_{i+1} \otimes V_i \otimes \cdots \otimes V_k.$$

Thm (Jimbo, '86): The alg. gen'd by the R_i is isom. to $\mathcal{H}_B$ (for GL_k), and the U and $\mathcal{H}_B$ actions are mutual centralizers, so we the decomp.

$$V^{\otimes k} = \bigoplus_{\lambda \vdash k} L_q^\lambda \otimes S_q^\lambda,$$

where L_q^λ, S_q^λ are irred, and deformations of the L^λ, S^λ.

Remarks

1) Jimbo's results helped kick-start huge breakthroughs. One notable example: Jones' Field Medal work on knot invariants.

2) This section only holds for GL_n, not a reductive group of any other type.

3) Not surprising that $U_q(\mathfrak{gl}_n)$ is in S-W duality w/ a deformation of $\mathbb{C}[S_n]$, but it is remarkable that this deformation turned out to be the Hecke alg.

4) I am not aware of any "natural" reason for remark 3, and in light of remark 2), might be hard to have a general result. Would be very interesting if such a result existed!

$$U(\mathfrak{sl}_2) = \langle e, f, h \mid [e,f] = 2h, \; \dots \rangle$$

$$e \mapsto E, \quad f \mapsto F, \quad h \mapsto K, K^{-1}, \quad 2h \leftarrow q^{\square}$$