Crystal Bases

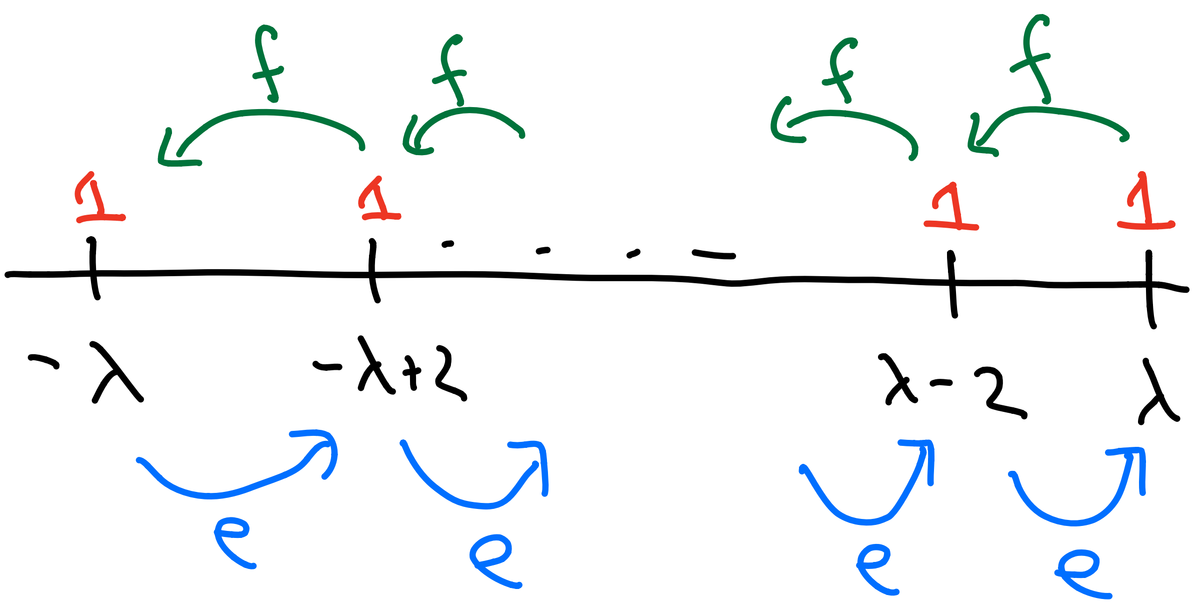
Motivation: $\mathfrak{sl}_2$-rep'ns

$\mathfrak{sl}_2 = \langle e, f, h \mid [e,f] = h,\ [e,h] = 2e,$
$[f,h] = -2f \rangle$

Highest wt. theory for $\mathfrak{sl}_2$:

Given λ: nonneg integer,
$V(\lambda)$: $\mathfrak{sl}_2$-irrep w/ highest wt. λ

Wt. space decomp:



Thus, let

$v_\lambda \in \lambda$-weight space of $V(\lambda)$

Then,

$$\mathcal{B} = \{ f^i v_\lambda \mid i = 0, \dots, \lambda \}$$

is a basis of $V(\lambda)$, and

$$e \cdot f^i v_\lambda = \left(-i^2 + (\lambda+1) i\right) f^{i-1} v_\lambda$$

So when we apply raising and lowering operators to elements of $\mathcal{B}$, we get (scalar multiples of) elements of $\mathcal{B}$ back.

Def: Let Φ be a root system with index set I and weight lattice Λ. A (seminormal) crystal of type Φ is a nonempty set $\mathcal{B}$ with maps:

$$e_i, f_i : \mathcal{B} \to \mathcal{B} \sqcup \{0\}$$

$$\mathrm{wt} : \mathcal{B} \longrightarrow \Lambda$$

such that if $x, y \in \mathcal{B}$, then

$$e_i(x) = y \iff f_i(y) = x.$$

If the above holds, then,

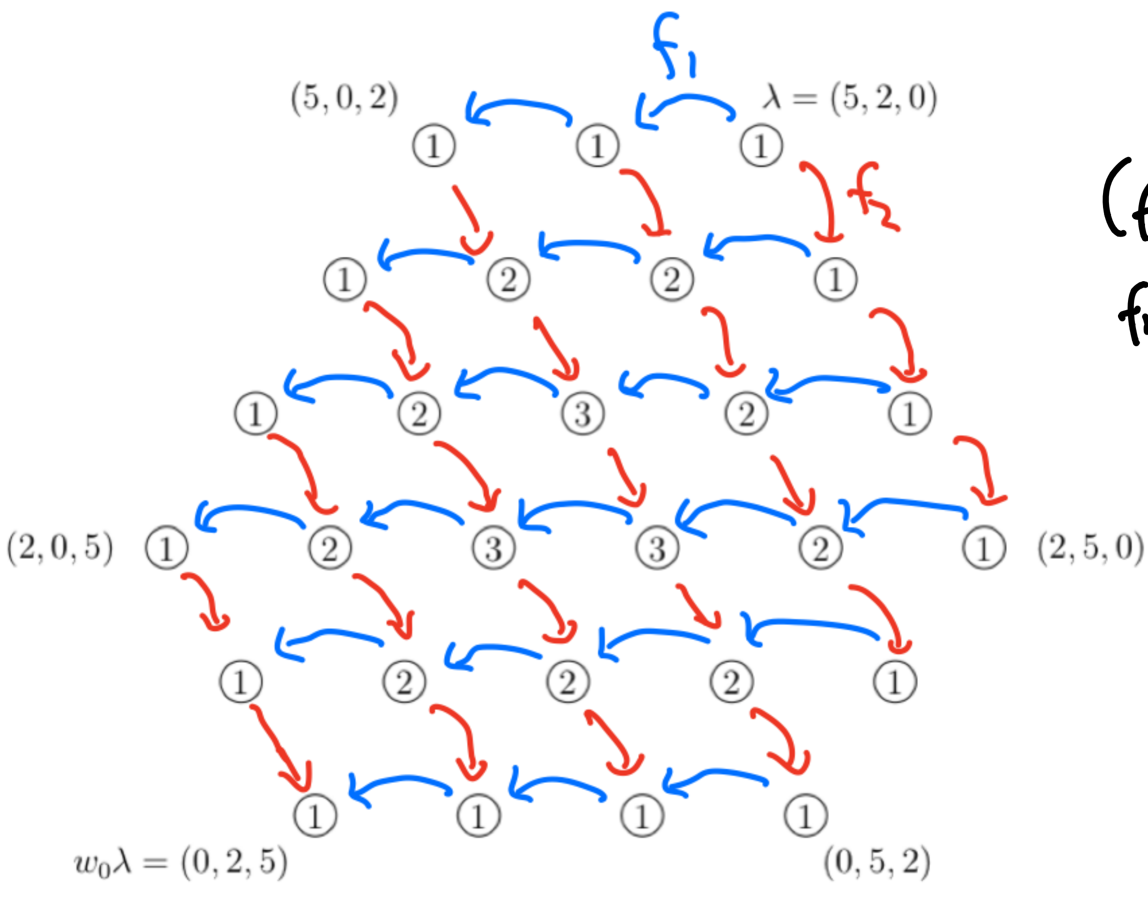
$$\mathrm{wt}(y) = \mathrm{wt}(x) + \alpha_i.$$

Now consider an arbitrary Lie algebra:

$$\mathfrak{g} = \langle e_i, f_i, h_i \mid \text{relations} \rangle.$$

Now, multiple raising/lowering operators & wt. spaces w/ dim > 1, so <u>no crystal bases!</u>

Ex: $\mathfrak{g} = \mathfrak{sl}_3$



(figure from Bump-Schilling)

Let's pass to quantum groups:

$$U_q(\mathfrak{g}) = \langle E_i, F_i, K_i, K_i^{-1} \mid \text{relations} \rangle$$

$q \neq 0$ & q not a root of unity.

<u>Good news</u>: The (type $\mathbb{1}$) rep'n theory of $U_q(\mathfrak{g})$ is closely related to that of $\mathfrak{g}$. All f.d. repns are <u>semisimple</u>, and for every dominant $\lambda \in \Lambda$, there is an irrep $L(\lambda)$ of $U_q(\mathfrak{g})$ with the same weights and weight space dimensions as the corresponding irrep. $V(\lambda)$ of $\mathfrak{g}$.

<u>Bad news</u>: Still no crystal basis!

Fix (Kashiwara): Let "$q \to 0$".

<u>Rough sketch</u>: Given $L(\lambda)$,

1) Define normalized/uniformized raising/lowering operators $\tilde{E}_i$, $\tilde{F}_i$, such that, for a given i, $\tilde{E}_i$, $\tilde{F}_i$, take us between basis vectors.

2) Construct an A-module $\mathcal{L}(\lambda)$ s.t. $\mathcal{L}(\lambda) \otimes_A k \cong L(\lambda)$, where A is a particular DVR called an "admissible lattice" that allows us to "set $q=0$".

3) Set $\overline{L}(\lambda) := \mathcal{L}(\lambda) / q\mathcal{L}(\lambda)$.

$\overline{L}(\lambda)$ has the same weight space structure as $L(\lambda)$ (and $V(\lambda)$).

4) Let $\overline{v}_\lambda$: image of v_λ in $\overline{L}(\lambda)$.

5) Set $\mathcal{B} := \{\underbrace{\tilde{F}_{i_1}\cdots\tilde{F}_{i_k}}_{\text{all combinations}}\overline{v}_\lambda\} \setminus \{0\}$

Thm (Kashiwara): $\mathcal{B}$ is a crystal basis for $\overline{L}(\lambda)$!

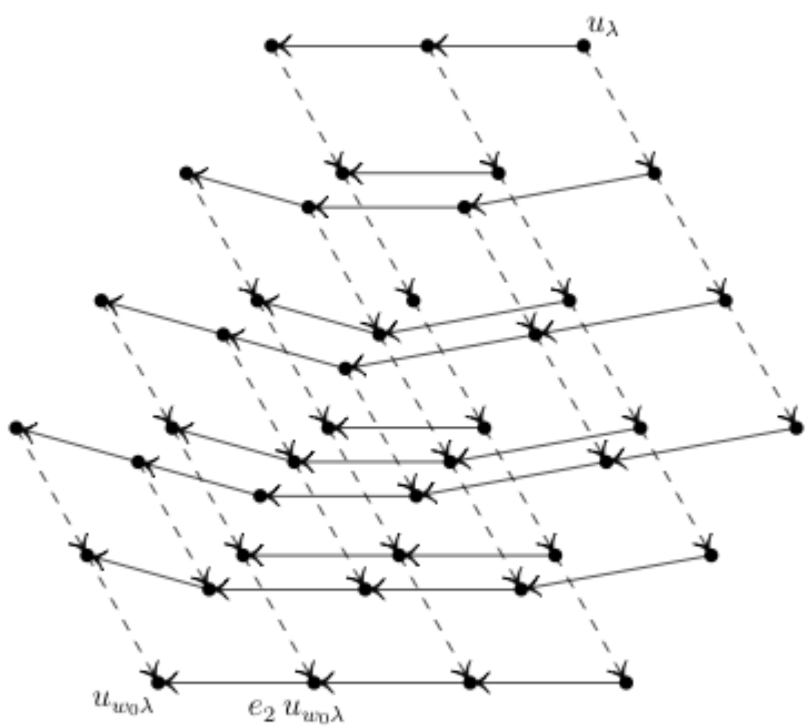


Fig. 4.1 The A_2 crystal with highest weight $\lambda = (5, 2, 0)$.

(figure from Bump-Schilling)

Want to emphasize: The crystal $\mathcal{B}$ also gives us a basis for $L(\lambda)$, $V(\lambda)$, although it's not a crystal basis for those modules.

Example: type A – crystals of tableaux.

Fact: In type A, $V(\lambda)$ has basis the set of semistandard Young tableaux (SSYT) of shape λ.

Even better: These SSYT form a crystal:

$\mathcal{B}(\lambda) := \{\text{SSYT of shape } \lambda\}$

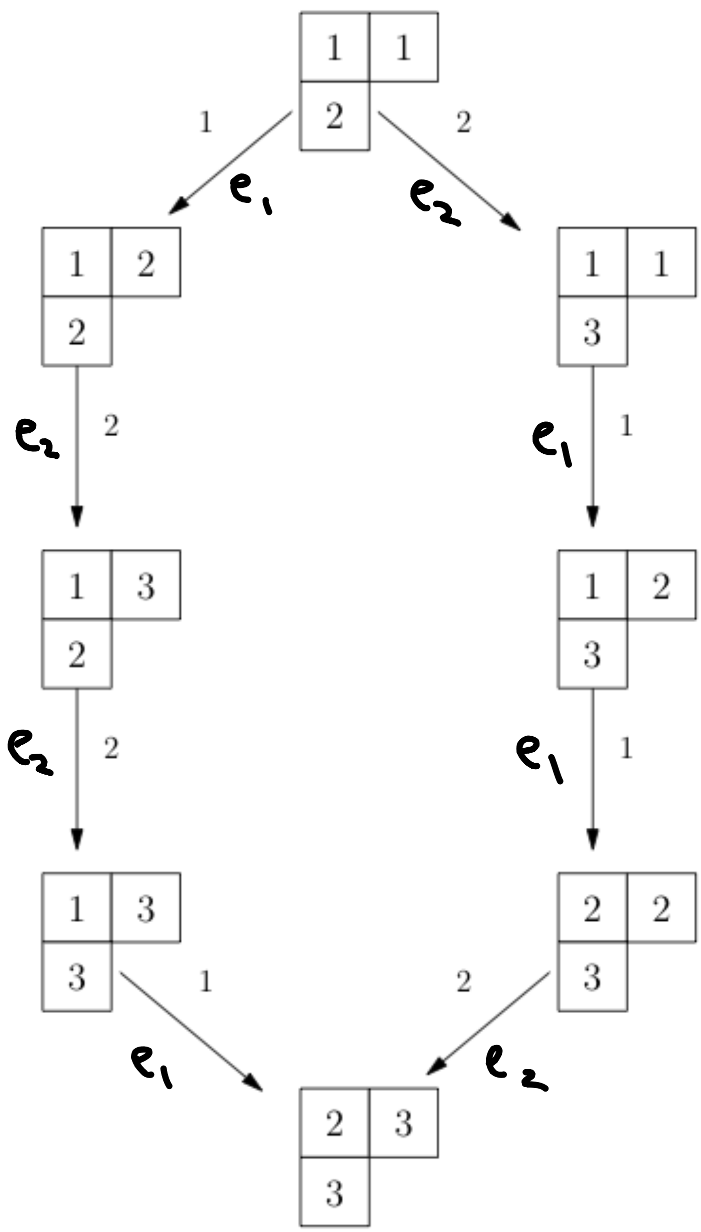
$wt(T) = z_1^{\#1's} \cdots z_n^{\#n's}$

$e_i(T) = T$ with first i set to $i+1$, reading from right to left, top to bottom

$f_i(T) = T$ with first $i+1$ set to i,
reading from right to left, top to bottom

Ex: $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = (2,1)$

$\mathcal{B} =$



(figure from Bump-Schilling)

Universal Highest Weight Crystal

Let λ, ν be dominant weights

Let $U^- := U_q(\underbrace{n^-}_{\{F_1, \dots, F_n\}})$

$$U^-_{-\nu} := \{u \in U^- \mid K_\mu u K_\mu^{-1} = q^{(\mu,\nu)} u \quad \forall \mu \in \mathbb{Z}\Phi\}.$$

Consider the map

$$\begin{aligned} U^-_\nu &\xrightarrow{\phi} \overline{L}(\lambda)_{\lambda-\nu} \\ u &\longmapsto u v_\lambda \end{aligned}$$

If $\lambda - \mu$ dominant, ϕ is <u>bijective</u>

Furthermore,

$$L(\infty)_{-\nu} := \phi^{-1}\left(\overline{L}(\lambda)_{\lambda-\nu}\right)$$

is independent of λ.

Let $B(\infty)_{-\nu}$ be the pullback of $\underbrace{B(\lambda)_{\lambda-\nu}}_{\{x \in B(\lambda) \mid wt(x) = \lambda - \nu\}}$ via this map

And let $B(\infty) = \bigsqcup_{\nu \text{ dom.}} B(\infty)_{-\nu}$

Then, for arbitrary dominant λ, we obtain a map

$$\overline{\varphi}_\lambda : B(\infty) \longrightarrow B(\lambda) \sqcup \{0\},$$

which is <u>bijective</u> on

$\mathcal{B}(\infty) \setminus \ker \overline{\varphi_\lambda}$.

Last piece: $\mathcal{B}(\infty)$ is a "canonical basis" for U^- relative to the modules $\overline{L}(\lambda)$. But we want a canonical basis relative to the quantum group modules $L(\lambda)$.

Thm (Lusztig, Kashiwara):
For every $b \in \mathcal{B}(\infty)$, there exists an element $G(b) \in U^-$ s.t $G(b) \equiv b \pmod{q}$
and
1) $\{G(b) \mid b \in \mathcal{B}(\infty)\}$ is a basis of U^-

2) For any dominant λ, the map $u \mapsto u v_\lambda$ induces a bijection:

$$\{ G(b) \mid b \in B(\infty),\ G(b)v_\lambda \neq 0 \}$$

$$\updownarrow$$

$$\{\text{basis of } L(\lambda)\}$$

3) The same thing holds in the Lie algebra setting if we replace U^- with $U(n^-)$ and $L(\lambda)$ with $V(\lambda)$.

Sources:

Jantzen: Lectures on Quantum Groups

Bump-Schilling: Crystal Bases

Brubaker: course notes & videos from topics course