Solvable Lattice Models, Statistical Mechanics, and Symmetric Functions

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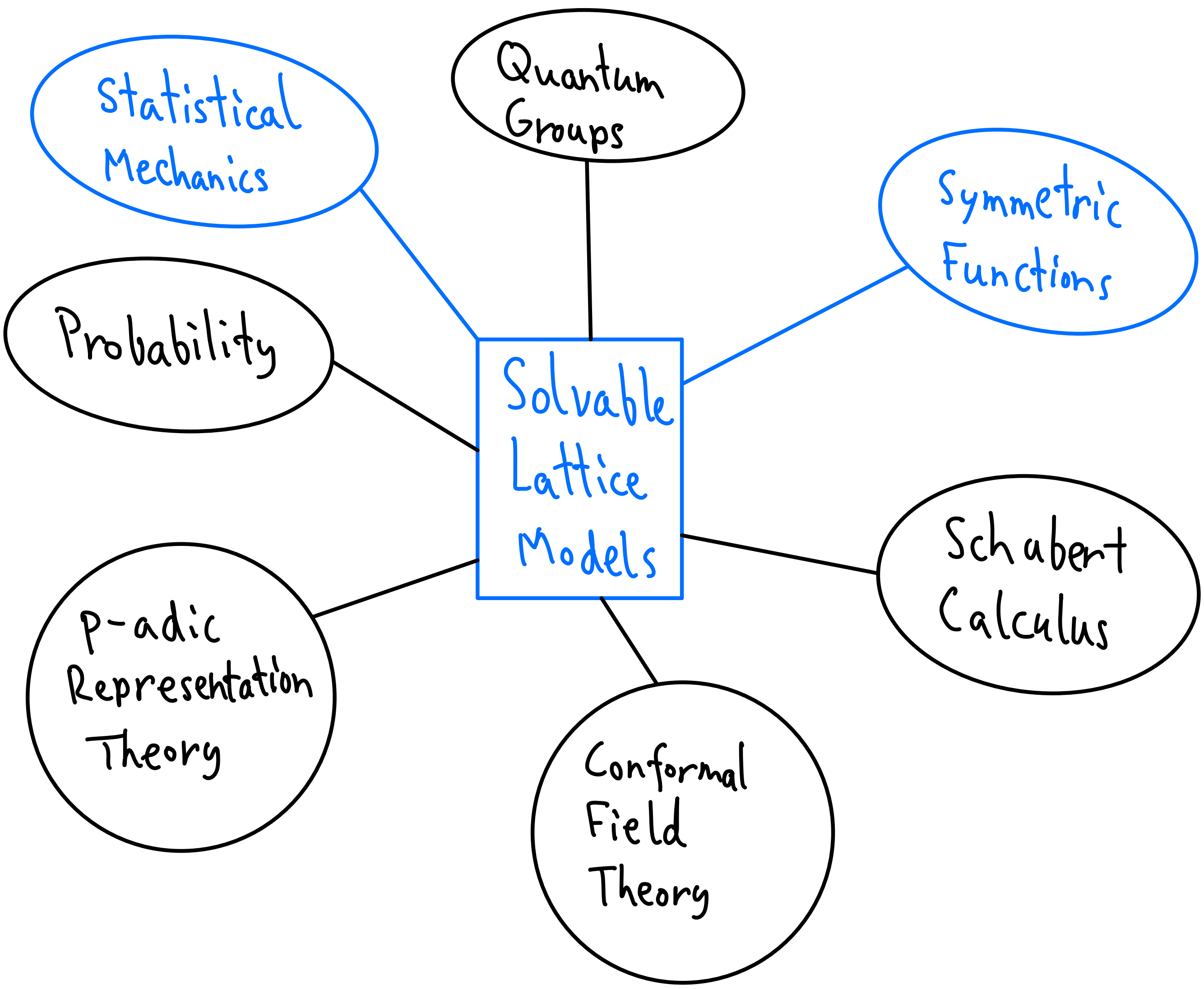
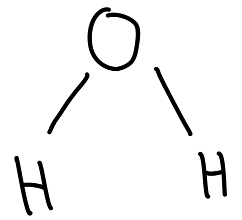
1) Square ice
2) Statistical mechanics and the partition function
3) Schur functions
4) The Yang-Baxter equation and symmetry
5) The branching rule for Schur functions
6) Bonus: Lattice models and ASMs



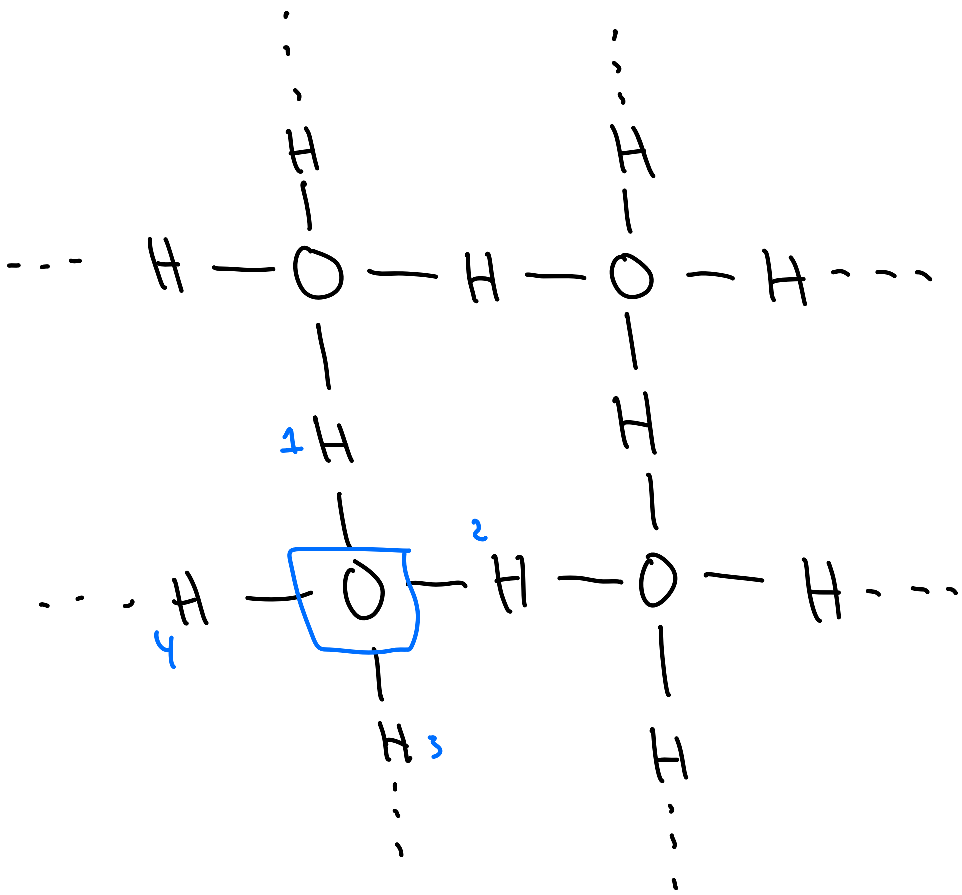
1) Square ice

One molecule of H_2O:



What about ice, where molecules have bonds to each other?

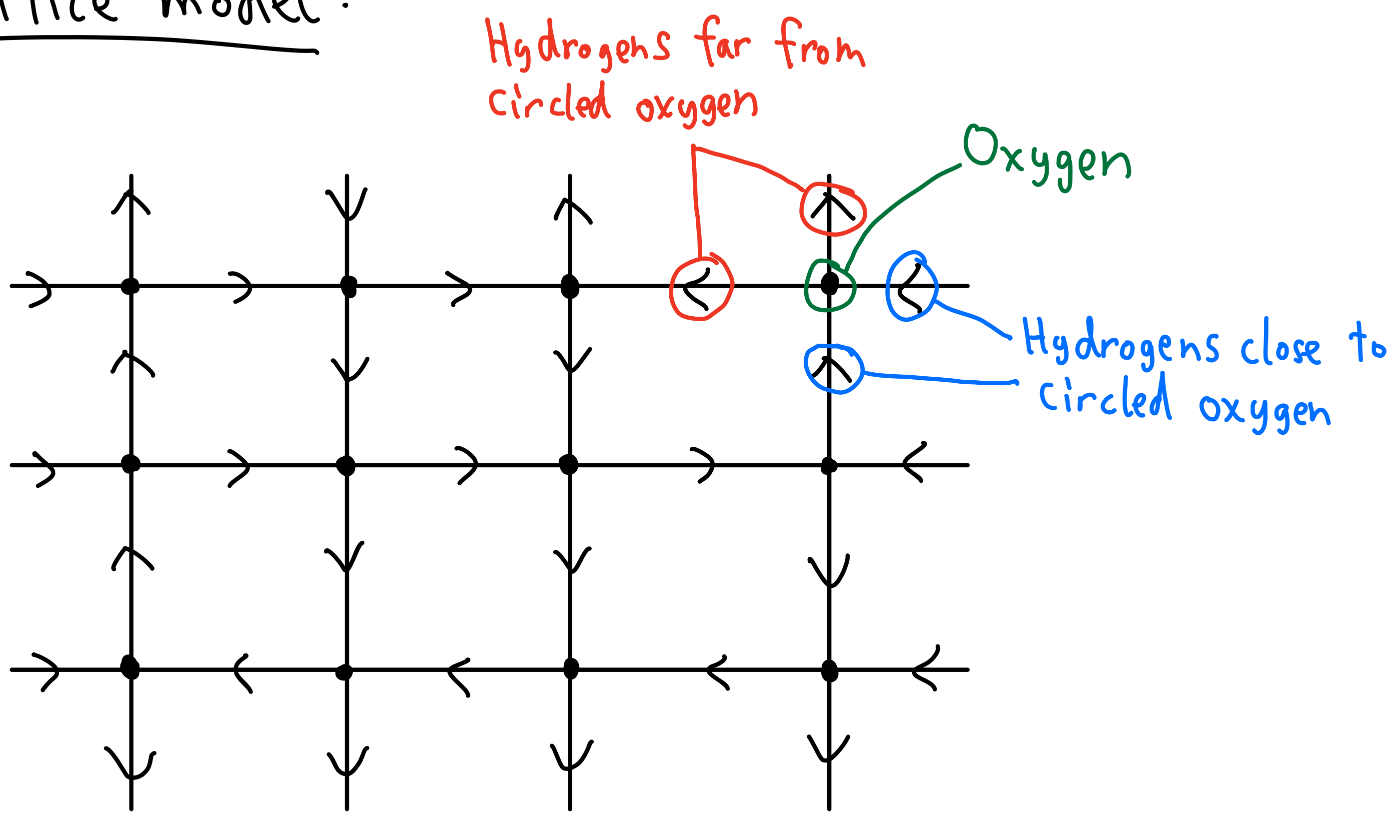
Square ice:



How come this is H_2O?

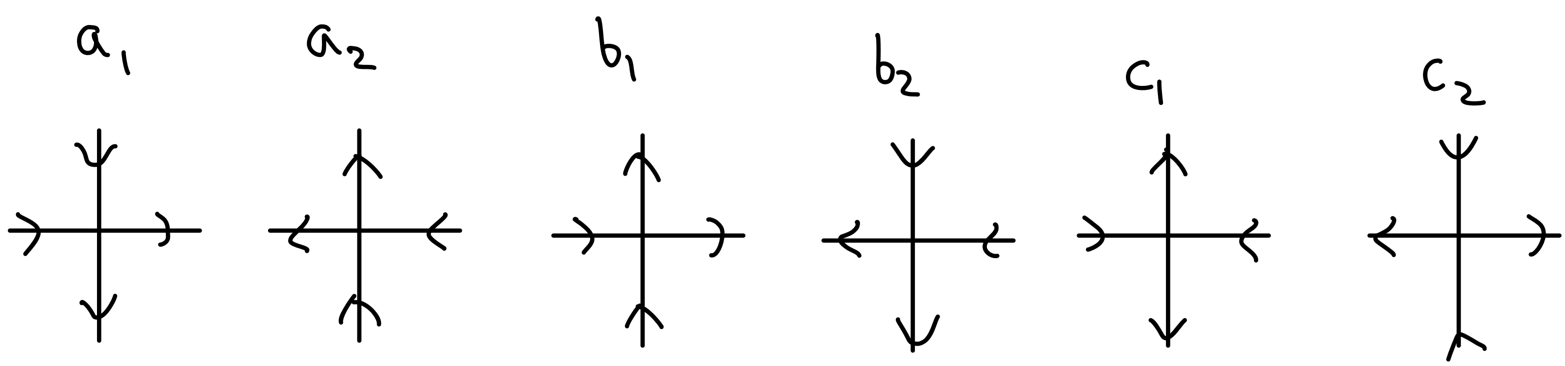
Answer: each hydrogen is "close" to one of its two adjacent oxygens

Lattice model:



Ice rule: around each vertex, two arrows point in and two point out.

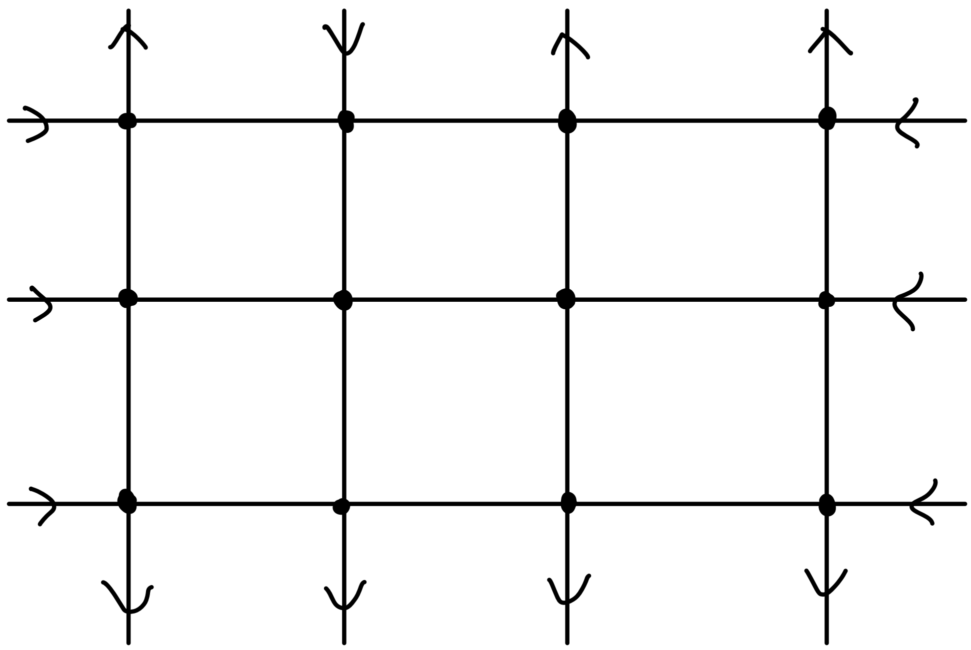
Six possibilities for each vertex:



2) Statistical mechanics and the partition function

Goal of statistical mechanics: understand global (macroscopic) properties of a system in terms of local (microscopic) interactions

Fix boundary conditions:



State: "Admissible" configuration of all the internal edges

{ each vertex must satisfy the ice rule

$$\text{Energy of state} = \sum_{\text{vertices}} \text{Energy of vertex}$$

Probability of a given state is inversely related to it's energy:

$$P(s) = \frac{e^{-\frac{1}{kT} \cdot \text{Energy}(s)}}{Z}$$

k: Boltzmann's constant

T: temperature

normalization factor ("partition function")

$$Z = \sum_{\text{state } s} e^{-\frac{1}{kT} \cdot \text{Energy}(s)}$$

$$= \sum_{\text{state } s} \prod_{\text{vertex } v} \underbrace{e^{-\frac{1}{kT} \cdot \text{Energy}(v)}}_{\text{Boltzmann weight of } v}$$

Since the partition function Z relates to probability, it is key when calculating the average of any quantity (energy, entropy, etc.)

If we can "compute Z efficiently", we call the system solvable.

For actual ice, we have:

$$wt(a_1) = wt(a_2) = wt(b_1) = wt(b_2) = wt(c_1) = wt(c_2) = 1$$

(see bonus section for connection with alternating sign matrices)

But for now, we'll keep the Boltzmann weights arbitrary...

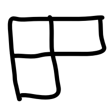
3) Schur functions

Most important "symmetric functions"

(useful, for instance in studying representation theory of $GL_n(\mathbb{C})$)

Tableau formula: $$\underset{\substack{\uparrow \\ \text{partition}}}{s_\lambda} = \sum_{T \in \underbrace{SSYT}_{\text{"semi-standard Young tableaux"}}} x^T$$

Ex: $\lambda = (2,1,0)$ $n = 3$

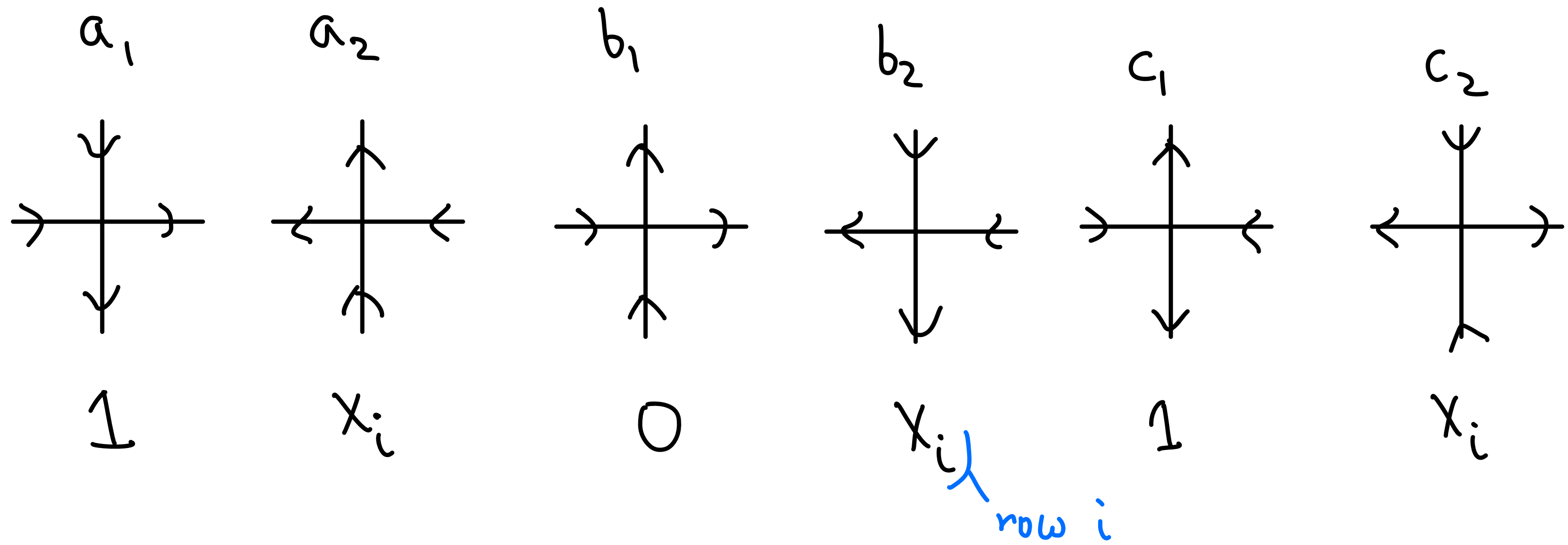


Tableaux: $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$

$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}$ } Symmetry is not obvious

$$s_\lambda(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2 x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2$$

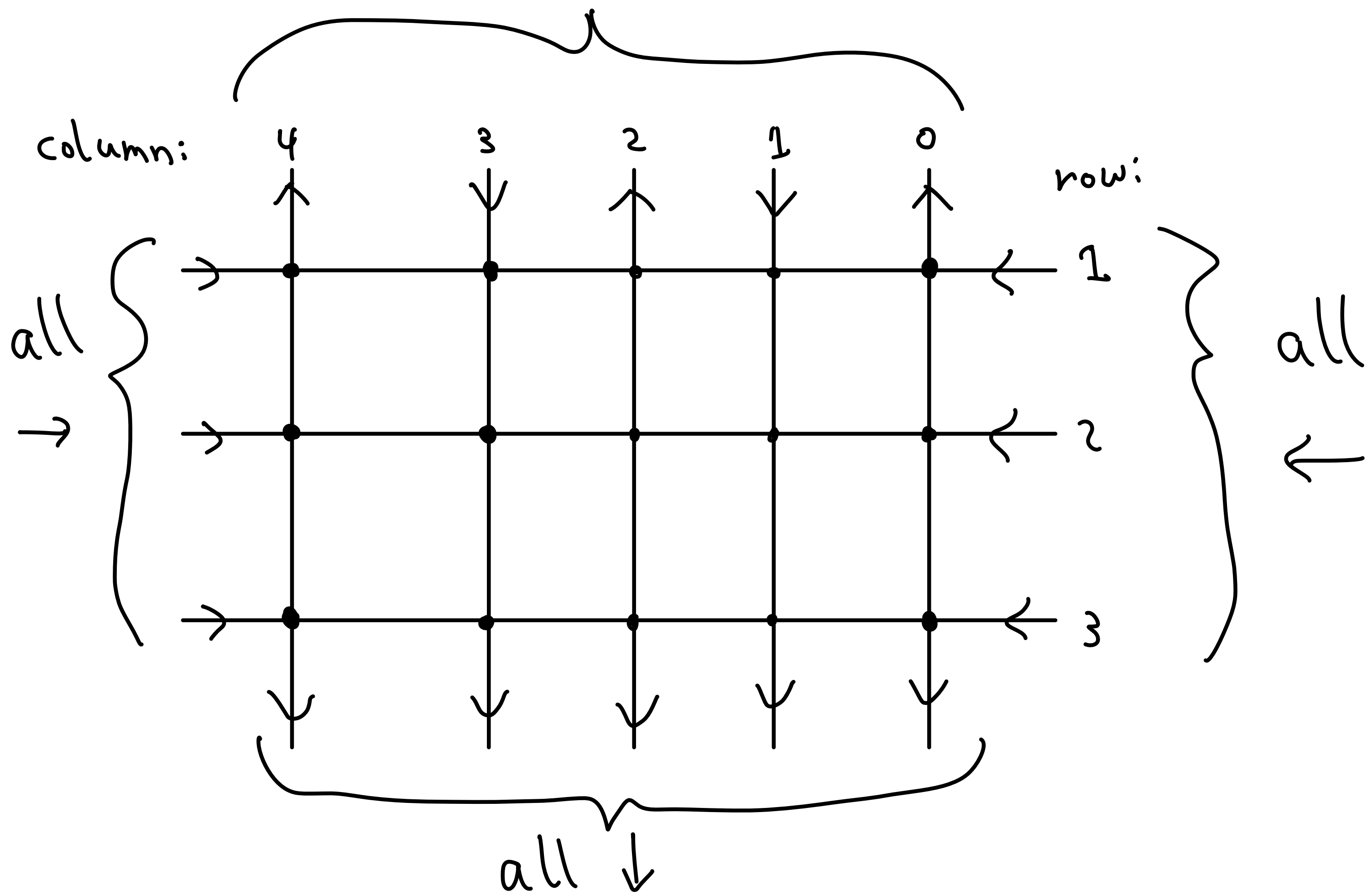
Now we'll define a lattice model for Schur polynomials:



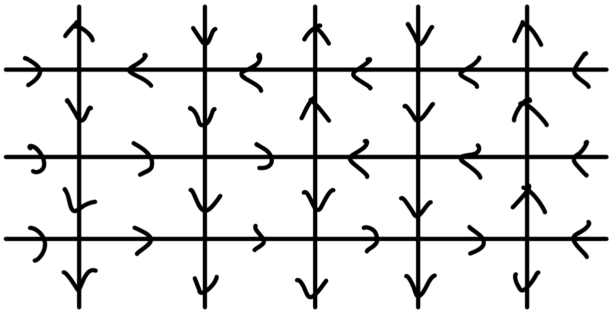
Boundary conditions

$\uparrow$ on parts of $\lambda + \rho = (4,2,0)$ $\big(\rho = (n-1, \dots, 0)\big)$

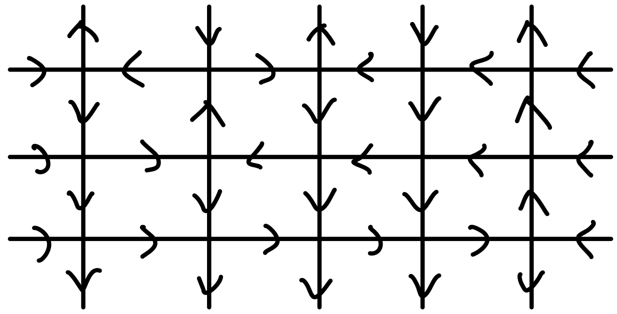
$\downarrow$ otherwise



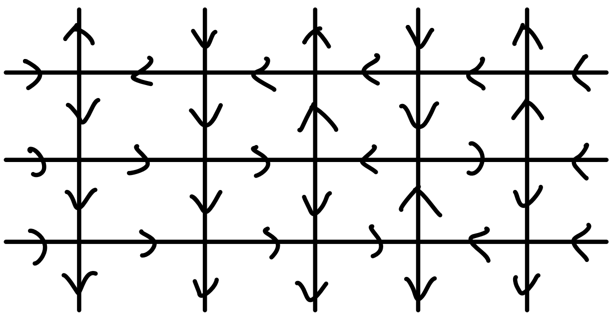
Admissible states:



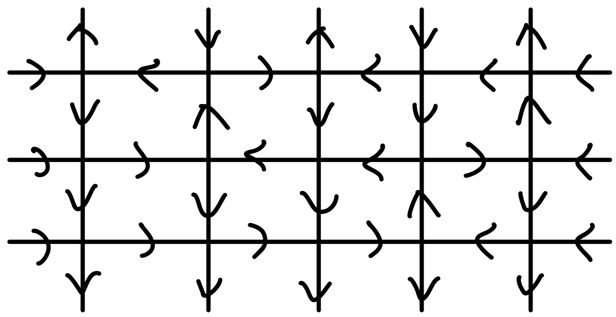
$x_1^4 x_2^2$



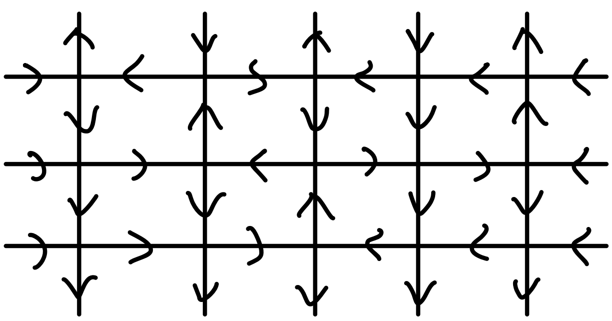
$x_1^3 x_2^3$



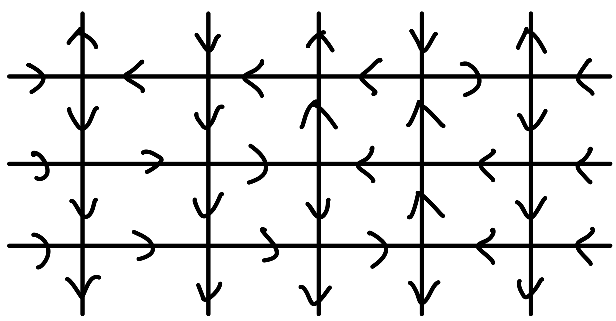
$x_1^4 x_2 x_3$



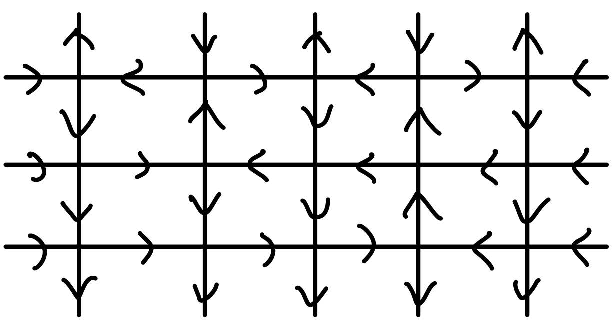
$x_1^3 x_2^2 x_3$



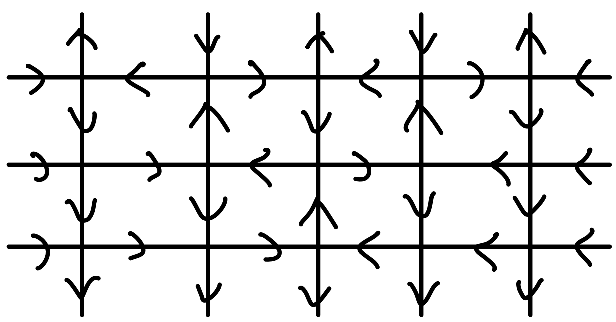
$x_1^3 x_2 x_3^2$



$x_1^3 x_2^2 x_3$



$x_1^2 x_2^3 x_3$



$x_1^2 x_2^2 x_3^2$

$$Z = x_1^4 x_2^2 + x_1^3 x_2^3 + x_1^4 x_2 x_3 + x_1^3 x_2^2 x_3$$
$$+ x_1^3 x_2 x_3^2 + x_1^3 x_2^2 x_3 + x_1^2 x_2^3 x_3 + x_1^2 x_2^2 x_3^2$$

$$= x_1^2 x_2 \cdot \left(x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + 2 x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 \right)$$

$$= \underbrace{x_1^2 x_2}_{\text{easy extra factor}} \cdot\ s_\lambda(x_1, x_2, x_3) \qquad (!)$$

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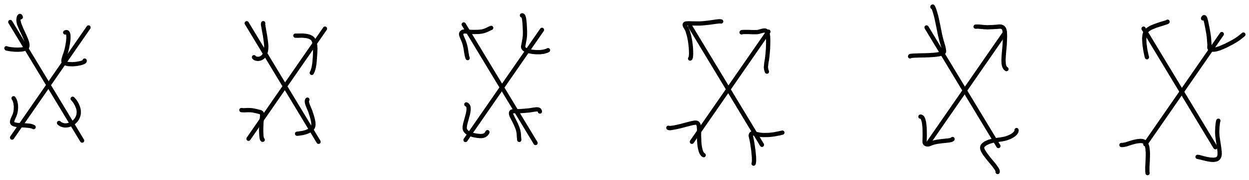
4) The Yang-Baxter Equation and Symmetry

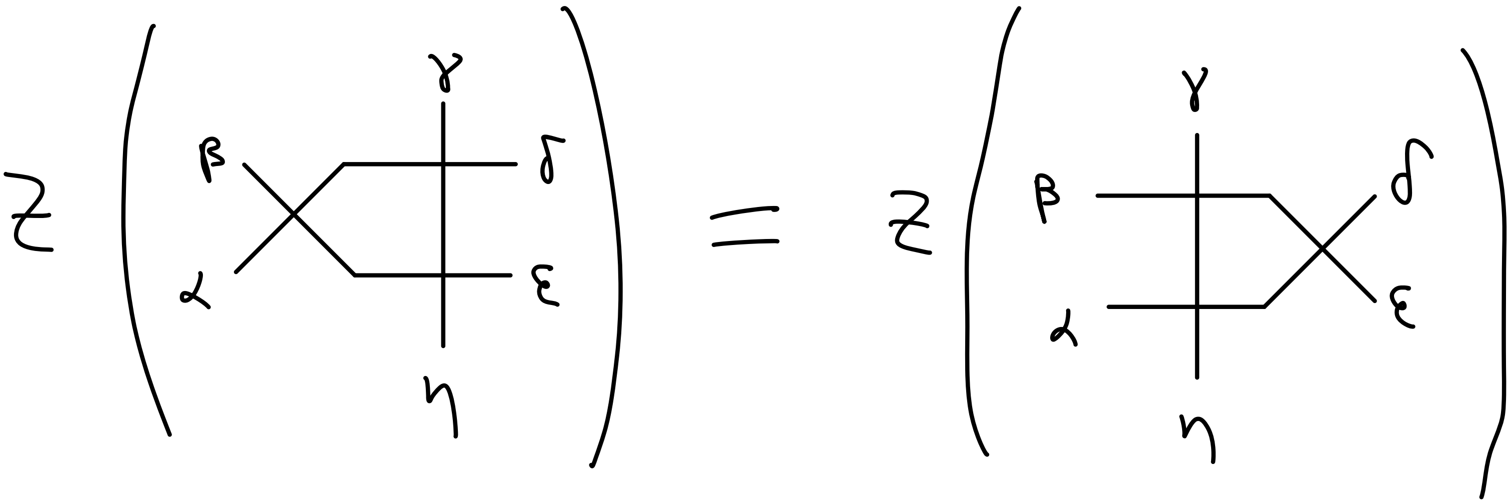
Main way to interpret "solvability":

Do these Boltzmann weights have a solution to the <u>Yang-Baxter Equation</u> (YBE)?

YBE solution: set of <u>diagonal</u> Boltzmann weights



Such that

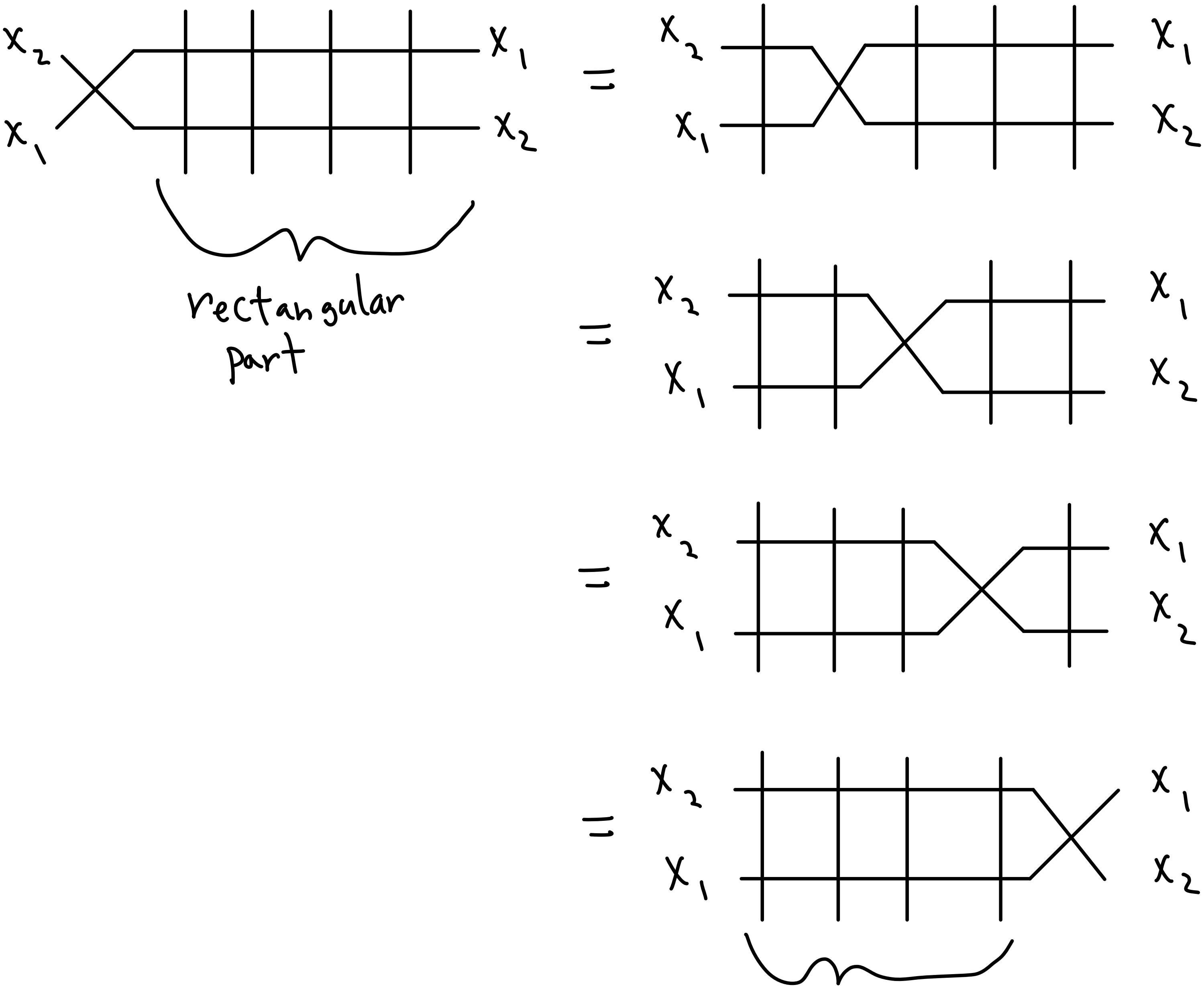


For <u>all</u> sets of boundary conditions $\alpha, \beta, \gamma, \delta, \varepsilon, \eta$.

Essentially, we can replace the diagram on the left with the diagram on the right whenever we want

But so what?

"Train argument":



rectangular part

same rectangular part, except with row parameters switched!

Upshot: The existance of a YBE allows us to determine symmetry properties

In particular, we can prove that Schur polys. are symm. functions!

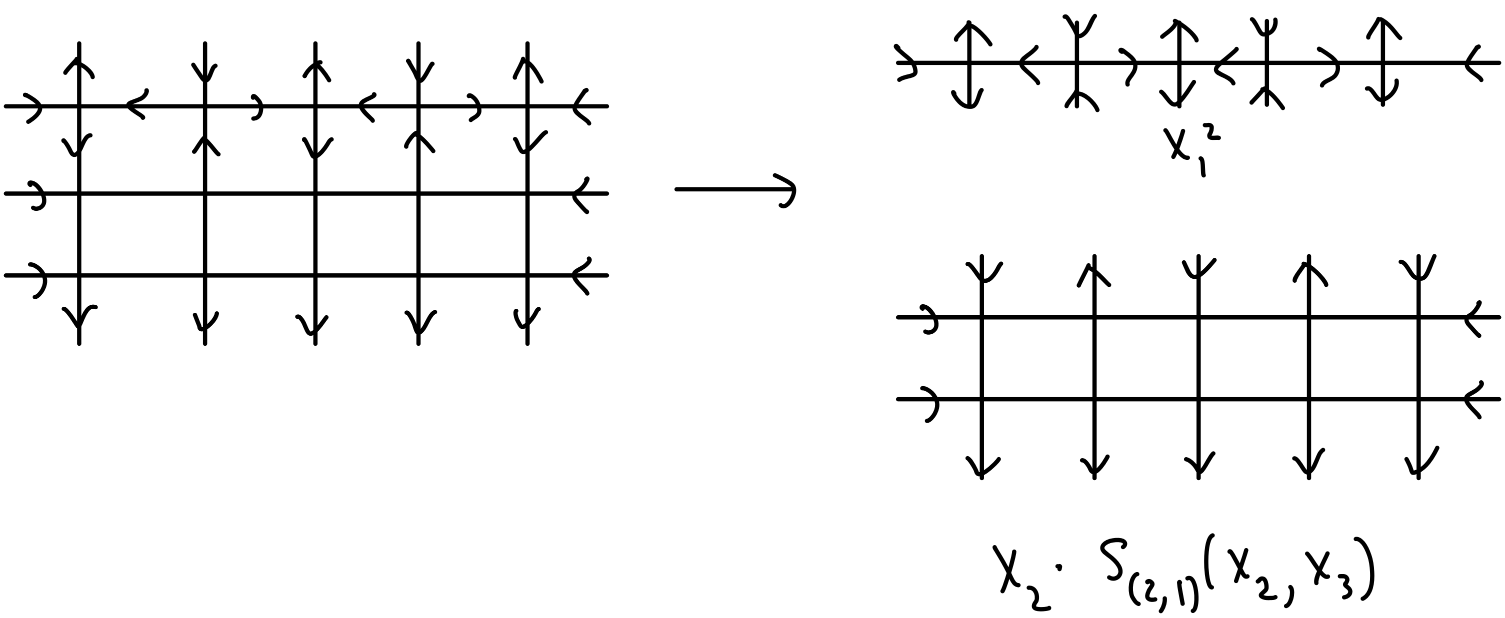
5) The branching rule for Schur functions

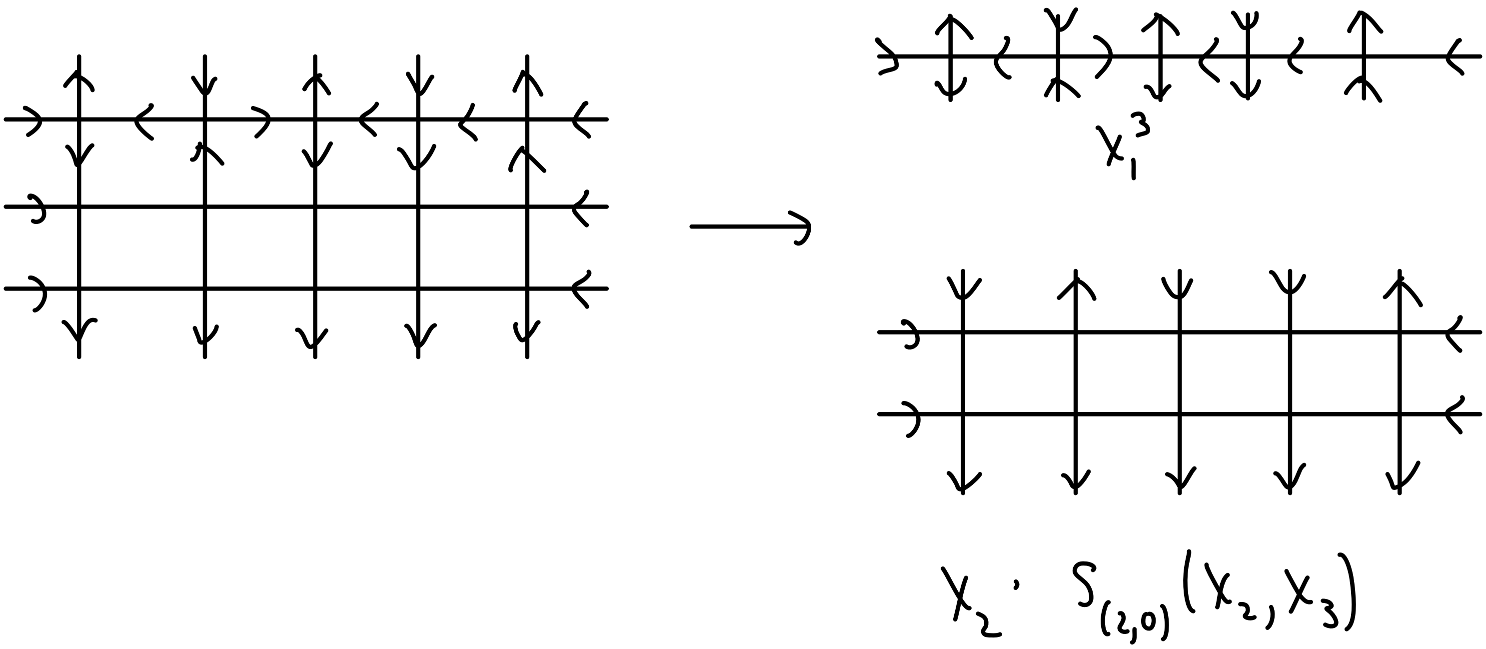
Branching rule: recursive formula for the Schur function s_λ in terms of smaller partitions μ

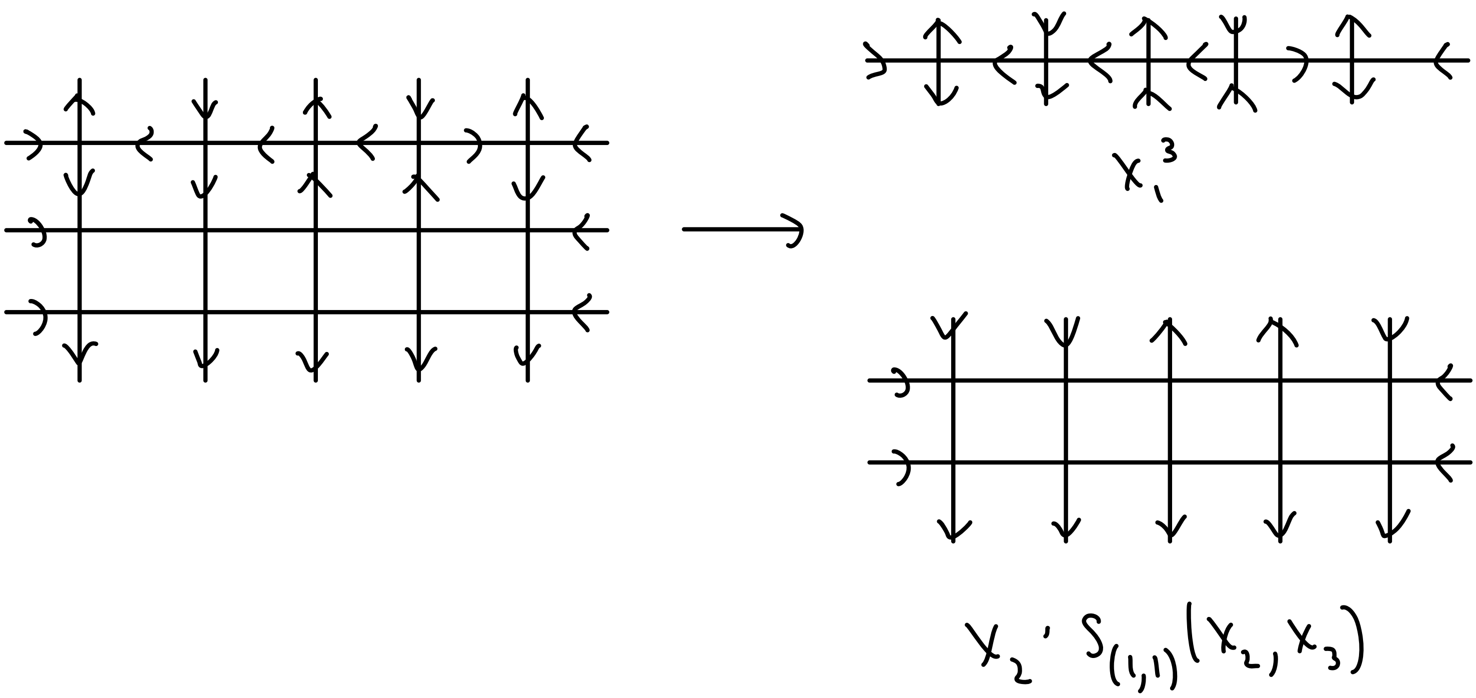
$$s_\lambda = \sum_\mu C_\mu^\lambda s_\mu$$

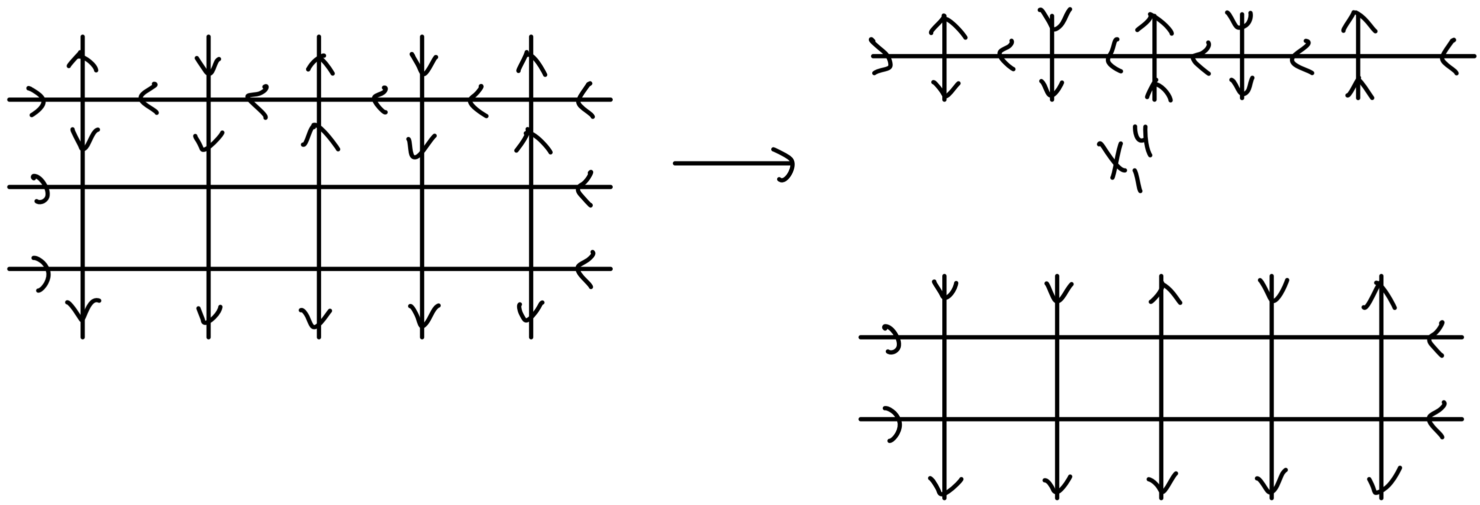
In lattice model land: remove top row

Ex: $\lambda = (2,1,0)$




x_1^3
$x_2 \cdot s_{(2,0)}(x_2, x_3)$


x_1^3
$x_2 \cdot s_{(1,1)}(x_2, x_3)$



$x_2 \cdot s_{(1,0)}(x_2, x_3)$

Final result:

$$s_{(2,1,0)}(x_1, x_2, x_3) = s_{(2,1)}(x_2, x_3) + x_1 \cdot s_{(2,0)}(x_2, x_3)$$

$$+ x_1 \cdot s_{(1,1)}(x_2, x_3) + x_1^2 \cdot s_{(1,0)}(x_2, x_3)$$

Exercise: Find the pattern. What is the general branching rule for $s_\lambda(x_1, \dots, x_n)$?
(Pieri's Formula)

This is only a small part of a v. large picture

- Can generalize to other Boltzmann weights
  - Other symmetric functions
- Can prove other identities (e.g. (dual) Cauchy identity)
- Can look at other types of lattice models

Much is unknown!

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6) Bonus: Lattice models and ASM's

Alternating-sign matrix:

Square $n \times n$ matrix such that:

- All entries are 0, 1, or -1
- In any row/column, the 1 and -1 entries alternate
- The first/last nonzero entry in each row/column is 1
  (i.e. the rows/colums sum to 1)

Relate to plane partitions, Dodgson condensation

ASM "Conjecture":

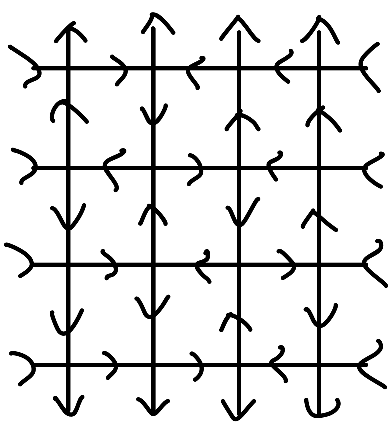
The number of $n \times n$ ASMs is:

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

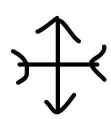
Proved by Zeilberger (1995) – 84 pages, computer proof

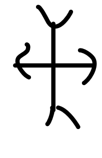
Kuperberg (1996) – 22 pages, including proofs of other conjectures

Bijection between ASMs and states of a lattice model:



$$\longleftrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

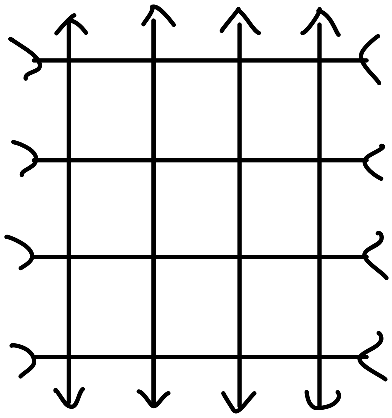
 $\longleftrightarrow$ 1

 $\longleftrightarrow$ -1

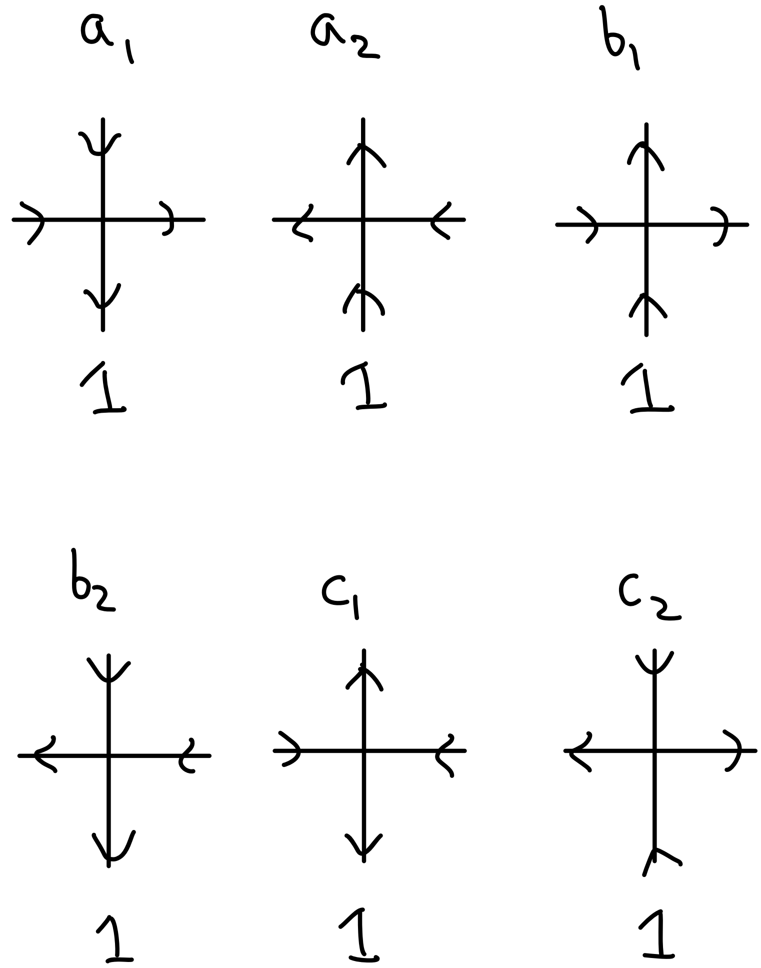
other $\longleftrightarrow$ 0

So, number of $n \times n$ ASMs = partition function of the following lattice model

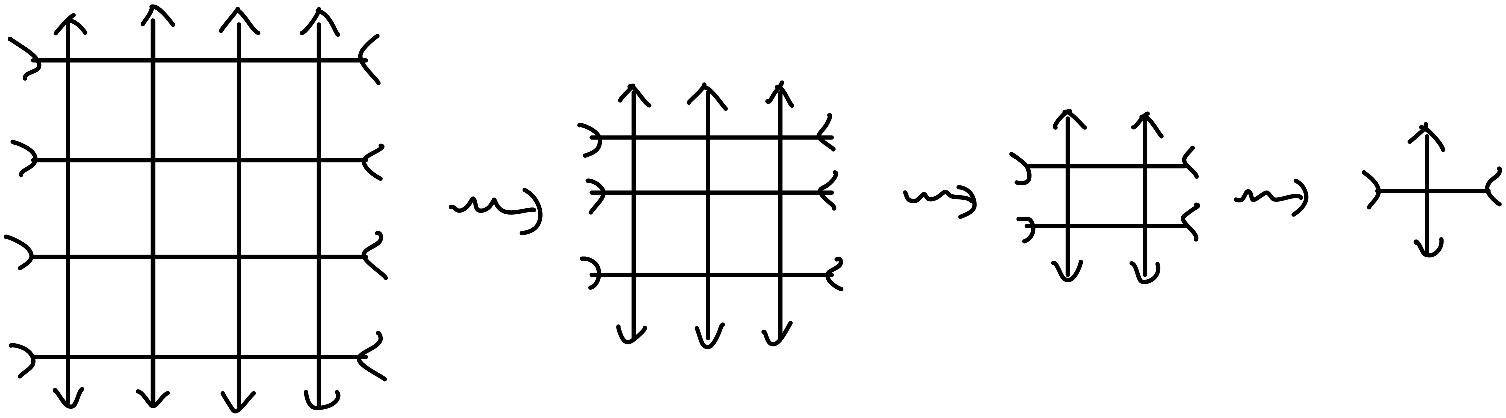
Boundary conditions



Boltzmann weights



Proof strategy: use recursion (Izergin-Korepin procedure) to reduce to smaller arrays, and prove via induction!



Fascinating that the Boltzmann weights for counting ASMs precisely match those for physical ice...