What is the Largest (Finite) Number?

1) Knuth up-arrow notation
2) Conway chained arrow notation
3) Fast growing hierarchy
4) TREE(n)
5) The philosophers ruin everything

Ground Rules:
1) Well-defined
2) Don't need efficient way to compute
3) Need to be able to prove it's large

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1) Knuth's up-arrows

First: towers of exponents.

$$3^3 = 27$$

$$3^{3^3} = 3^{(3^3)} = 3^{27} \approx 7 \text{ trillion}$$

$$10^{100} = 1 \text{ googol}$$

$$10^{10^{100}} = 1 \text{ googolplex}$$

$$\left.3^{3^{3^{3^{3}}}}\right\}\ 5 \text{ copies}$$

$$\left.3^{\cdot^{\cdot^{3}}}\right\}\ 100 \text{ copies}$$

Write $a \uparrow b := a^b$

$$a \cdot b = \underbrace{a + \cdots + a}_{b}$$

$a \uparrow b = a^b = \underbrace{a \cdot \cdots \cdot a}_{b}$

so

$a \uparrow\uparrow b := \underbrace{a \uparrow (a \uparrow \cdots (a \uparrow a))}_{b} = a^{\cdot^{\cdot^{a}}}\Big\}b$

$3 \uparrow\uparrow 3 = 3^{3^3} = 3^{27}$

$\underline{3} \uparrow\uparrow 5 = 3^{3^{3^{3^3}}}$ (more than 3^{trillion} digits)

Let's do 3 arrows; here's where it gets crazy

$a \uparrow\uparrow\uparrow b := \underbrace{a \uparrow\uparrow (a \uparrow\uparrow \cdots (a \uparrow\uparrow a))}_{b}$

$3 \uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow (3 \uparrow\uparrow 3) = 3 \uparrow\uparrow (3^{27}) = 3^{\cdot^{\cdot^{3}}}\Big\}3^{27}$

$\underline{3} \uparrow\uparrow\uparrow 4 = 3 \uparrow\uparrow (3 \uparrow\uparrow (3 \uparrow\uparrow 3))) = 3 \uparrow\uparrow 3^{\cdot^{\cdot^{3}}}\Big\}3^{27}$

4 arrows: $3 \uparrow\uparrow\uparrow\uparrow 3 = 3 \uparrow\uparrow\uparrow (3 \uparrow\uparrow\uparrow 3)$

$= 3 \uparrow\uparrow\uparrow \left(3^{\cdot^{\cdot^{3}}}\Big\}3^{27}\right)$

Further:

$3\uparrow^{5} 3$

$3\uparrow^{10^{100}} 3$

$3\uparrow^{3\uparrow\uparrow\uparrow\uparrow 3} 3$

$g_1 = 3\uparrow^{4} 3$

$g_2 = 3\uparrow^{g_1} 3$

$\vdots$

$g_{64} = 3\uparrow^{g_{63}} 3$ ← Graham's number

2) Conway's chained arrows

e.g. $2 \to 3 \to 4 \to 5 \to 6$

Rules

p, q: integers, X: chain

1) The chain p is just the number p

2) $p \to q = p^q$

3) $X \to 1 \to \ldots = X$

4) $X \to (p+1) \to (q+1) = X \to (X \to p \to (q+1)) \to q$

Turns out: $p \to q \to r = p \uparrow^{r} q$

Example:

$$3 \to 3 \to 3 = 3 \to (3 \to 2 \to 3) \to 2$$
$$= 3 \to (3 \to (3 \to 1 \to 3) \to 2) \to 2$$
$$= 3 \to (3 \to 3 \to 2) \to 2$$
$$= 3 \to (3 \to (3 \to 2 \to 2) \to 1) \to 2$$
$$= 3 \to (3 \to (3 \to (3 \to 1 \to 2) \to 1) \to 2$$
$$= 3 \to (3 \to (3 \to 3)) \to 2$$
$$= 3 \to 3^{3^3} \to 2$$

Four $a \to b \to c \to d$ captures #'s like Graham's #.

$3 \to 3 \to 64 \to 2 <$ Graham's # $< 3 \to 3 \to 65 \to 2$

$3 \to 3 \to 3 \to 3$: far far bigger

$3 \to 3 \to 3 \to 3 \to 3$ etc.

$3 \to^{10} 3$

$3 \to^{3 \to 3 \to 3} 3$

etc.

3) Fast-growing hierarchy

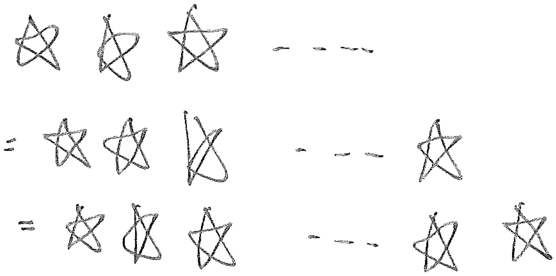
Puts recursive properties like this roughly "in bijection" with "large ordinals".

$f_\alpha(n)$: n integer, α: large ordinal

ω = ----

$\omega + 1$ = ---

$\omega + 2$ = ---



$f_\omega(n) > 2\uparrow^{n-1} n$

$f_{\omega^2}(n) > n \rightarrow^n n$

$f_{\omega^\omega}(10)$: unbelivably large number

Different types of recursion $\leftrightarrow$ different things we can do to ω.

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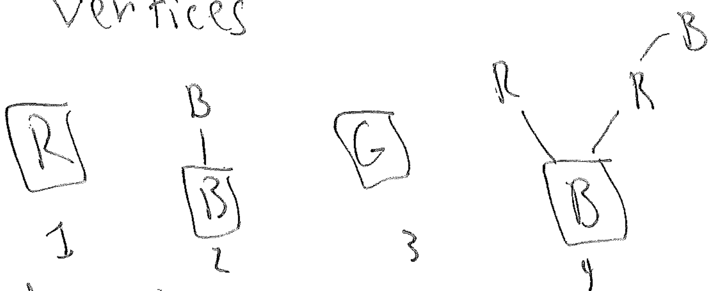
4) TREE(n)

Let's play a game: Take n colors and draw a sequence of (rooted) trees, where the i-th tree can have at most i vertices

e.g. n=3



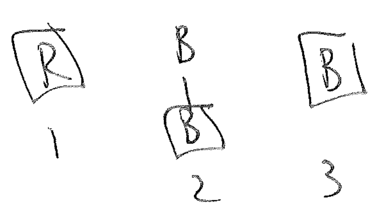
Game ends when a tree "contains" an earlier one.

TREE(n): longest possible game w/ n colors

~~Kruskal~~ Theorem (Kruskal): TREE(n) always finite

TREE(1) = 1

TREE(2) = 3



TREE(3): far bigger than anything we've seen so far

5) The Philosophers Ruin Everything

Not going to talk about: Busy Beaver function

Rayo's number (defined in a "Big Number Duel")

Rayo(10^{100}), where Rayo(n) = largest # definable ~~using ≤n~~ using "first-order set theory" in ~~≤10^100~~ n symbols.

Two ~~ways~~ reasons this goes crazy

1) Can define TREE, then write TREE(TREE(⋯ (TREE(3)))

2) You're allowed to define a bunch of stuff, and then say "largest number in ≤ TREE(3) symbols using stuff we've defined"

Craziest thing is the most obvious: all of these numbers are finite! Any other science can't ever deal with something like Graham's number. But for math, of course, any proof "for all n" automatically works for all these numbers, no problem. This remarkable power is what makes mathematice unique, special, and utterly baffling.