Alcove Walks

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Plan: We will use the combinatorial method of alcove walks to understand geometrically-interesting "cells" of matrix groups. (Intersection $U^{-}vI \cap IwI$ of double cosets)

Part I: The algebra

1) The flag variety

A Lie group is a group that is also a $\underbrace{\text{manifold}}_{\text{(locally like Euclidean space)}}$.

- They're everywhere
  (connections to nearly every area of math & physics)
- Most Lie groups are matrix groups
  e.g. GL_n, SL_n, SO_n, Sp_n, over $\mathbb{R}$ or $\mathbb{C}$
- Beautiful, detailed structures

Miracle: much of the structure holds over any field ("Chevalley Groups")

For today: $G = SL_n$
(Let's agree that some def's & all examples will have $G = SL_3$)

Let B be the subgroup of upper triangular matrices (Borel subgroup):

$$B = \begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$$

Quotient G/B: flag variety

A flag is a sequence of subspaces

$$\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$$

where $\dim V_i = i$.

Flag variety: one of the most important objects in algebra

However: B is not normal,

so G/B is not a group!

Brilliant "fix": instead of left cosets, let's consider double cosets.

Given $g \in G$, $BgB = \{g' \in G \mid g' = b_1 g b_2,\ b_1, b_2 \in B\}$.

Double cosets are disjoint, so we can write:

Bruhat decomposition: $G = \bigsqcup_{w \in \underbrace{W}_{\text{set of representatives}}} BwB$

Key fact: Turns out W is a group, called the Weyl group for G.

(For $G = SL_n$, $W = S_n$).

So, $G/B = \bigsqcup_{w \in W} \underbrace{BwB/B}_{\text{union of left } B \text{ cosets}}$

Upshot: every element gB of G/B corresponds to a unique $w \in W$ and a (usually nonunique $b \in B$): $gB = bwB$

#coolconnection to Sunita's project: membership in double Bruhat cells BwB gives a criterion for total positivity!

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2) The affine flag variety

Going to step it up!

Field has been arbitrary up to now, but from now on, let

$$G = SL_n(F), \text{ where } F = \mathbb{C}((t))$$

F is the fraction field of $\mathcal{O} = \mathbb{C}[[t]]$.

$\mathcal{O}$ has unique maximal ideal (t), and there is a map $\mathcal{O} \to \mathbb{C}$ setting $t=0$.

e.g. $1+2t+3t^2+4t^3+\dots \mapsto 1$

This induces a map $SL_n(\mathcal{O}) \xrightarrow{\phi} SL_n(\mathbb{C})$.

Iwahori subgroup:

$$I = \{ g \in SL_n(\mathcal{O}) \mid \phi(g) \in B \}.$$

$$I = \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ (t) & \mathcal{O} & \mathcal{O} \\ (t) & (t) & \mathcal{O} \end{bmatrix}$$

The affine flag variety is G/I.

Again, not a group, <u>but</u>:

Iwahori decomposition:

$$G = \bigsqcup_{w \in \tilde{W}} IwI,$$

and $\tilde{W}$ is a group, called the affine Weyl group.

Example: Let $g = \begin{bmatrix} 1/t & 2t & 2t^2 \\ & t & t^2 \\ & & 1 \end{bmatrix}$

Then $g \in B$, so

$$g = \underbrace{\begin{bmatrix} 1/t & 2t & 2t^2 \\ & t & t^2 \\ & & 1 \end{bmatrix}}_{\in B} \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{\in W} \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{\in B}.$$

$g \in B1B$

Also,

$$g = \underbrace{\begin{bmatrix} 1/t & 2t & \\ & t & \\ & & 1 \end{bmatrix}}_{\in B} \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{\in W} \underbrace{\begin{bmatrix} 1 & & \\ & 1 & t \\ & & 1 \end{bmatrix}}_{\in B}$$

$g \in B1B$

Notice that the elements of W are the same.

Now, $g \notin I$, but

$$g = \underbrace{\begin{bmatrix} 1 & 2 & 2t^2 \\ & 1 & t^2 \\ & & 1 \end{bmatrix}}_{\in I} \underbrace{\begin{bmatrix} t^{-1} & & \\ & t & \\ & & 1 \end{bmatrix}}_{\in \tilde{W}} \underbrace{\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}_{\in I}$$

Now, let's explore $W, \tilde{W}$ ...

3) Weyl group & affine Weyl group

Let $G = SL_3$, so $W = S_3$, $\widetilde{W} = \widetilde{S}_3$

Note that $s_1 = (12)$, $s_2 = (23) \in S_3$ have order 2.

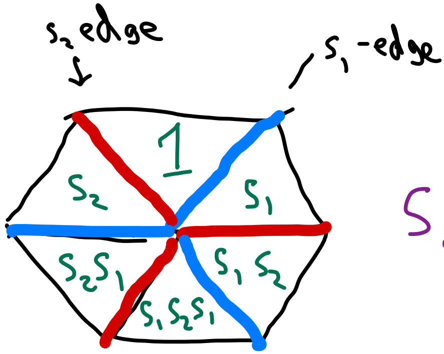
$$S_3 = \left\langle s_1, s_2 \;\middle|\; s_1^2 = s_2^2 = 1,\ s_1 s_2 s_1 = s_2 s_1 s_2 \right\rangle$$

(braid rel'n)

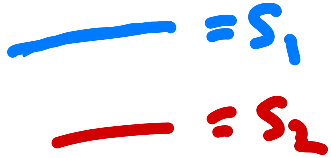
(Coxeter presentation)

Pictorially:

$\triangle$ = alcove

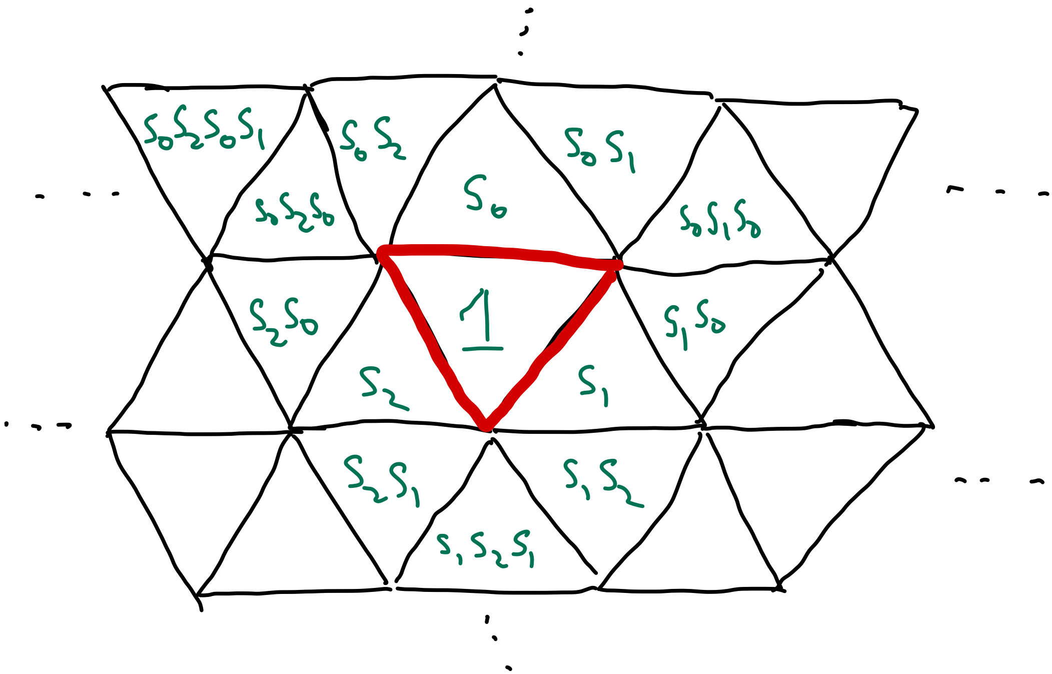


$S_3 \longleftrightarrow \{\text{alcoves}\}$



Similarly,

$$\widetilde{S}_3 = \left\langle s_0, s_1, s_2 \;\middle|\; s_0^2 = s_1^2 = s_2^2 = 1,\ \begin{array}{l} s_0 s_1 s_0 = s_1 s_0 s_1 \\ s_0 s_2 s_0 = s_2 s_0 s_2 \\ s_1 s_2 s_1 = s_2 s_1 s_2 \end{array} \right\rangle$$



$$\tilde{S}_3 \longleftrightarrow \{\text{alcoves}\}$$

REU Exercise 7.1

a) Write out all 6 elements of S_3 as minimal length products of s_1, s_2. What is special about (13)? ~~$s_1 s_1$~~

b) Prove that S_3 bijects with the alcoves in the first diagram.

c) Prove that $\tilde{S}_3$ bijects with the alcoves in the second diagram. You just proved that $\tilde{S}_3$ is infinite!

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4) Steinberg generators

First another decomposition:

Let $U^- = \begin{bmatrix} 1 & & \\ * & 1 & \\ * & * & 1 \end{bmatrix}$.

Then,

$$G = \bigsqcup_{w \in \tilde{W}} U^- w I$$

← affine Weyl group

Let's get more precise information about the elements of U^-, I, $\tilde{W}$

Steinberg generators:

$$\underset{\underset{x_1(c)}{\shortparallel}}{X_{\alpha_1}(c)} = \begin{bmatrix} 1 & c & \\ & 1 & \\ & & 1 \end{bmatrix} \qquad X_{-\alpha_1}(c) = \begin{bmatrix} 1 & & \\ c & 1 & \\ & & 1 \end{bmatrix}$$

$$\underset{\underset{x_2(c)}{\shortparallel}}{X_{\alpha_2}(c)} = \begin{bmatrix} 1 & & \\ & 1 & c \\ & & 1 \end{bmatrix} \qquad X_{-\alpha_2}(c) = \begin{bmatrix} 1 & & \\ & 1 & \\ & c & 1 \end{bmatrix}$$

$$\underset{\underset{x_0(c)}{\shortparallel}}{X_{\alpha_0}(c)} = \begin{bmatrix} 1 & & \\ & 1 & \\ ct & & 1 \end{bmatrix} \qquad X_{-\alpha_0}(c) = \begin{bmatrix} 1 & & ct^{-1} \\ & 1 & \\ & & 1 \end{bmatrix}$$

Let $n_i(c) := x_i(c)\, x_{-\alpha_i}(-c^{-1})\, x_i(c)$,

$n_i := n_i(1)$, $\qquad h_i(c) = n_i(c)\, n_i^{-1}$

REU Exercise 7.2:

a) Show that $x_i(c_1)x_i(c_2) = x_i(c_1+c_2)$

b) Compute $n_i, h_i(c)$, $i = 0,1,2$

Which of the x_α, n_i, h_i are in U^-?

Which are in I?

c) Prove that (up to flipping signs)

n_0, n_1, n_2 satisfy the same relations as s_0, s_1, s_2

d) Solve the following equation for $i,j = 0,1,2$:

$$n_i^{-1}x_j(c) = x_?(?)\ldots x_?(?)n_i^{-1}.$$

e) Prove symbolically that if $c \neq 0$,

$$x_i(c)n_i^{-1} = x_{-\alpha_i}(c^{-1})x_i(-c)h_i(c)$$

(Main Folding Law)

f) Use parts d, e to show that

when $j \neq i$,

$$n_j^{-1}x_i(c)n_i^{-1} \in U^- n_j^{-1} I.$$

Part II: The alcove walk model

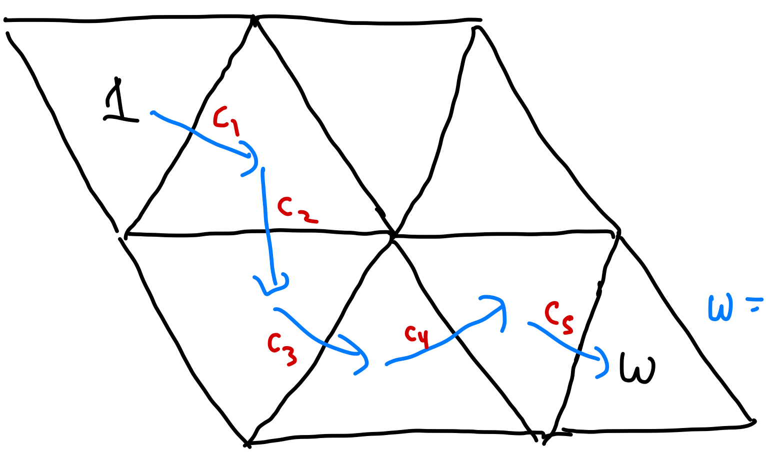
$$U^- v I = \{ \underbrace{x_{\gamma_1}(d_1) \cdots x_{\gamma_k}(d_k)}_{\in U^-} \underbrace{n_{j_1}^{-1} \cdots n_{j_k}^{-1}}_{v = s_{j_1} \cdots s_{j_k}} I \mid d_1, \ldots, d_k \in \mathbb{C} \}$$

$(v \in \tilde{W})$

Theorem 1 (Parkinson-Ram-Schwer '08):
Let $w = s_{i_1} \cdots s_{i_\ell} \in \tilde{W}$ be a reduced expression.
Then in G/I,

$$IwI = \{ x_{i_1}(c_1) n_{i_1}^{-1} \cdots x_{i_\ell}(c_\ell) n_{i_\ell}^{-1} I \mid c_1, \ldots, c_\ell \in \mathbb{C} \}$$

1) Alcove walks



$w = s_1 s_2 s_0 s_2 s_1$

(Labelled) alcove walk: A shortest path walk to w, where every edge is labelled by an element of $\mathbb{C}$.

Corollary (PRS '08):

$$I w I / I \longleftrightarrow \left\{ \begin{array}{l} \text{labelled alcove} \\ \text{walks from} \\ 1 \text{ to } w \end{array} \right\}$$

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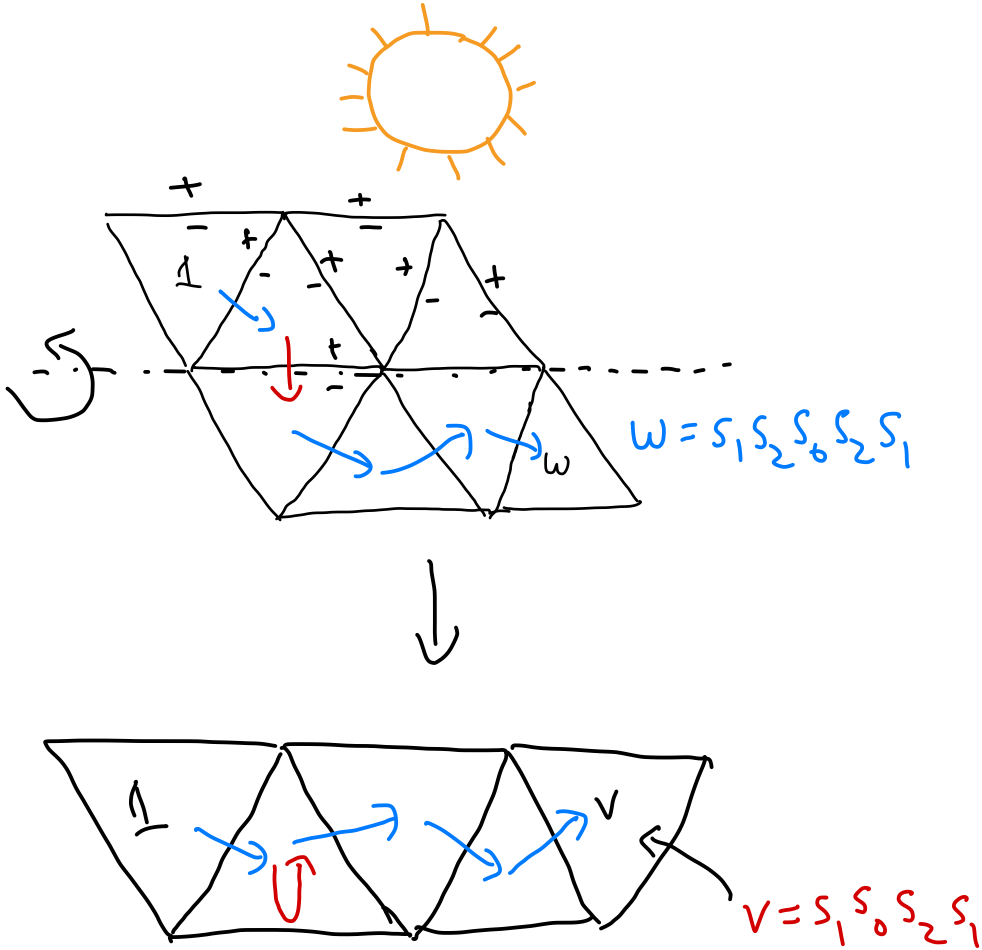
2) Folded alcove walks

Let the "sun" be at the top of the page. The positive side of each edge is the side that the sun hits.

We look at positively-folded alcove walks:

(edge-labels are implied)



This is a positively folded alcove walk of type w ending in v.

Theorem 2 (PRS '08): In G/I, there is a bijection:

$$\left(U^{-}vI \cap IwI\right)/I \longleftrightarrow \left\{\begin{array}{l}\text{labelled positively folded} \\ \text{alcove walks of type } w \\ \text{which end in } v\end{array}\right\}$$

Proof technique: Apply the main folding law repeatedly to an element of IwI.

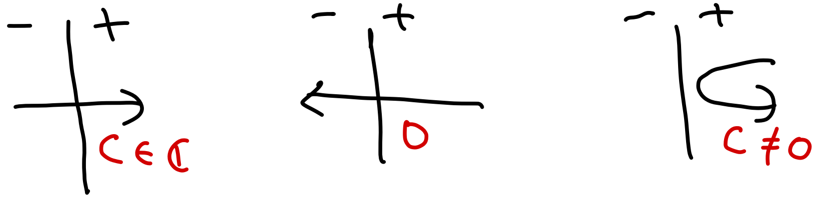
REU Exercise 7.3: Let $w = s_2 s_1 s_0 s_1 s_2$, $v = s_2 s_0 s_1 s_2$

(a) How many alcove walks of type w are there?

(b) Describe the elements of IwI. (Use Thm 1).

(c) How many positively folded alcove walks of type w ending in v are there?

(d) Describe the elements of $U^{-}vI \cap IwI$ using (b), (c), Thm 2, and the following label restrictions:



3) Triple intersections

Theorem 3 (PRS, Beazley–Brubaker):

a) $U^+ vI \cap IwI \leftrightarrow$ {labelled negatively folded alcove walks of type w ending in v}

$$U^+ = \begin{bmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{bmatrix}$$

b) The triple intersection

$U^- v_1 I \cap IwI \cap U^+ v_2 I \leftrightarrow$ {labelled positively folded alcove walks of type w ending in v_1 that correspond to negatively folded alcove walks ending in v_2.}

Theorem 4 (Beazley-Brubaker): When $G = SL_2$, the above bijection allows us to evaluate a certain number theoretic "special function" on SL_2 in terms of Gelfand-Tsetlin patterns. (#cool connection to Ben's project)

REU Problem 7: (Also: algebraic interpretation of the sam).

a) For $G = SL_3$, given $w, v_1, v_2 \in \tilde{W}$, when is $U^- v_1 I \cap I w I \cap U^+ v_2 I$ nonempty?

b) Figure out a combinatorial formula for its size (i.e. measure)

c) Can we do the same thing for other Chevalley groups (SL_4? SL_n? GL_n?), or for other double coset decompositions?

d) Can we use our results on triple intersections to compute certain special functions on G?