Finite Hecke Algebras and Their Characters

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Abstract

We explore some of the major results in the study of finite Hecke algebras and their character tables. These algebras are useful in the study of representations of finite Chevalley groups, and also appear in the study of quantum groups, and knot/link invariants. In Section 3, we see three different definitions for the Hecke algebra, define the character table, and state some character formulas. Then in Section 5, we prove Starkey's Rule, a computation of a Type A Hecke algebra character table, using the argument from [GP00], and use this argument to compute character values for certain elements in the Hecke algebra of Type B.

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1 Introduction

The Hecke algebra of a group G relative to a subgroup H is defined to be the algebra of H-biinvariant functions on G under convolution (and potentially some topological conditions) [Bum, p. 6]. These algebras are important in many different areas of math, including the representation theory of G, and early work was done by Iwahori, Matsumoto, and others (e.g. [Iwa64], [Mat64], [IM65]).

A prime motivation for the study of Hecke algebras is to study the representation theory of reductive groups. The Borel-Matsumoto Theorem [Bum, §1, §2] shows in various contexts that representations of G with H-fixed vectors correspond bijectively to representations of the Hecke algebra of G relative to H. In the case where G is a reductive group over a nonarchimedian local field and H is either the maximal compact subgroup or Iwahori subgroup of G, the associated Hecke algebra is reasonably nice, a boon for studying its representation theory. Since every smooth representation of G has a vector fixed by a compact open subgroup, it is theoretically possible to classify the representations of G through representations of various ascending Hecke algebras. The difficulty in this approach is that as H gets small, the presentation of the Hecke algebra becomes very complicated, and so the representation theory of the Hecke algebra is difficult.

This paper explores certain aspects of the representation theory of the "finite Hecke algebra", $\mathcal{H}$, where G is a finite Chevalley group and H is a Borel subgroup. This object $\mathcal{H}$ occurs in a surprising number of contexts. $\mathcal{H}$ can be written as a deformation of the group algebra of the Weyl group of G, and by Tits' deformation theorem, is abstractly isomorphic to that Weyl group. In type A, it turns out to be the correct "quantum" analogue of S_n in the context of Schur-Weyl duality. The HOMFLYPT polynomial is an invariant of knots and links, and can be interpreted as a trace function on the type A Hecke algebra. And through the Kazhdan-Lusztig basis, the Hecke algebra has applications to Schubert varieties and Verma modules.

The wide reach of the finite Hecke algebra is echoed in its variety of definitions. We look at three different definitions of the Hecke algebra in Section 3, and prove them all equivalent under certain conditions. The first definition of $\mathcal{H}$ is by generators and relations: as a deformation of the group algebra of a Coxeter group. The second definition is the one mentioned above, as the convolution algebra of Borel-biinvariant functions on a finite Chevalley group. The third definition, in type A only, is by Schur-Jimbo duality: $\mathcal{H}$ is the centralizer of the action of the quantum group $U_q(\mathfrak{gl}_n)$ in tensor powers of the standard representation of $U_q(\mathfrak{gl}_n)$. These different definitions aren't obviously equivalent; the fact that they are in fact the same speaks to the fundamental nature of the Hecke algebra.

We focus here on the generic Hecke algebra, where the parameters q_s are transcendental. This is for two reasons. First, the algebra is easier to study, since we don't need to worry about q_s being a root of unity (in which case $\mathcal{H}$ may not be semisimple). Second, keeping the parameters generic gives us the flexibility to specialize them on demand.

Because of their wide-ranging importance, the representation theory of finite Hecke algebras has been an object of study for half a century. By Tits' Deformation Theorem (Section 4.1), $\mathcal{H}$ is generically semisimple, and over a large enough field its representations correspond bijectively to the representations of the underlying Coxeter group W. These representations are easier to understand, so a useful way to study the representations of $\mathcal{H}$ is through the representations of W.

The character theory of the finite Hecke algebra has also been well-studied. Just defining an appropriate character table for a Hecke algebra takes some work, and we discuss this in Section 4.2. The irreducible Hecke algebra characters have been computed for every type (see [GP00, §9-11]), but a clean type-independent formula is not known. Instead, there are many results for different types with their own strengths and weaknesses. A main goal of this paper is to describe and compare some of these results.

We discuss three different character formulas/methods. The first method, Starkey's Rule, is known only in type A, and is the oldest character formula for Hecke algebras. It is the main focus of Section 5. The advantage of Starkey's rule is that it is a non-recursive combinatorial formula in terms of the character table of W. The second method is the Murnaghan-Nakayama rules for the classical types (A, B, and D), which are recursive formulas based off the classical Murnaghan-Nakayama rule for the characters of S_n. The third method is a set of linear relations given by Geck and Pfeiffer [GP00] that is particularly useful for the exceptional types.

Section 2 of this paper gives some representation theory preliminaries for Coxeter groups and semisimple algebras, and then Section 3 gives three equivalent definitions of the Hecke algebra. Then Section 4 discusses representations of Hecke algebras. Section 4.1 gives Tits' Deformation Theorem, which says that a Hecke algebra and its underlying Coxeter group have the same representation theory. Section 4.2 defines the character table of $\mathcal{H}$, and the rest of Section 4 gives three different methods of obtaining the character table of $\mathcal{H}$.

Starkey's Rule is a main focus of this paper, and Section 5 gives a (nearly) full proof of this result. Starkey's rule is a combinatorial formula that is straightforward to compute.

In fact, it is just matrix multiplication by a certain matrix. Its explicitness is an asset in practical use: it is much more straightforward to use in trace formula calculations than the other methods. As an example of this, in Section 5.4 we give a proof due to Geck and Jacon [GJ03] that uses Starkey's Rule to determine the weights of Ocneanu's trace, which arises in the study of the HOMFLYPT polynomial. Unfortunately, Starkey's rule only covers type A. We follow the Starkey's Rule proof in [GP00, § 9.2], but work to extend each lemma as much as possible to see if the result can be extended to other types. This results in an extension of Starkey's Rule to certain conjugacy classes in type B (see Proposition 5.13). It is still an open question whether a combinatorial formula in the vein of Starkey's Rule exists for the full type B Hecke algebra, or if results exist in type D or the exceptional types. This paper arose out of these attempts to bring Starkey's Rule to bear on other types.

2 Representation Theory Preliminaries

The Hecke algebra is a deformation of the group algebra of a Coxeter group, and therefore the representations of Hecke algebras has commonalities with those of Coxeter groups, or semisimple algebras in general. In this section we describe some basic results of representation theory, much of which will be useful in our study of Hecke algebras.

2.1 Finite Groups and Semisimple Algebras

We mostly follow Chapters 1-3 of [Web16] in this section. Let A be a unital associative algebra over a field k. A representation V of A is a unital left A-module. We assume that A is finite dimensional over k. Under this condition, the A-orbit of every vector is finite dimensional, so every irreducible representation of A is finite dimensional. Thus we will only consider finite dimensional representations.

An important question is whether every indecomposible A-module is irreducible. This is equivalent to A being a completely reducible module over itself, and we say that A is semisimple if this property holds.

Maschke's Theorem [Web16, Theorem 1.2.1] tells us that representations of a finite group G are completely reducible as long as $\mathrm{char}(k)$ doesn't divide $|G|$, and so in this case $k[G]$ is a semisimple algebra, since the categories *Rep*-G and *Rep*-$k[G]$ are equivalent.

By Schur's Lemma [Web16, Theorem 2.1.1], if k is algebraically closed and V, W are two irreducible representations of A, then $\mathrm{Hom}_A(V, W) = k$ if $V \cong W$ and 0 otherwise. If in addition A is semisimple, then by the Artin-Wedderburn Theorem [Web16, Theorem 2.1.3],

$$A \cong M_{n_1}(k) \oplus \cdots \oplus M_{n_r}(k)$$

as an A-algebra, where the n_i are the dimensions of the irreducible representations of A. This result leads to the formula that

$$\dim A = \sum n_i^2.$$

2.2 Characters

The character is an important invariant for any finite dimensional representation of a group or algebra. For a finite group, the idea of a character naturally leads to the notion of a character table, but a character table of a semisimple algebra is much harder to define in a useful way. The character table of a group has columns indexed by conjugacy class representatives, and in order to define the character table of an algebra, we must choose a set of "standard" elements to take their place.

Let $\chi(g) := \mathrm{Tr}(\rho(g))$ be the character of a representation ρ of a finite group G. χ is fixed by conjugation, so is constant on conjugacy classes of G. We then define the character table of G to be the following matrix: the rows are indexed by the irreducible characters of G, the columns are indexed by (representatives of) the conjugacy classes of G, and the table values are the character values at these representatives.

The character table of a finite group is an orthogonal square matrix [Web16, Theorems 3.2.3, 3.4.3], and thus the characters of G are an orthogonal basis for the space of class functions on G. In addition, the columns are an (orthogonal) basis for the same vector space, and we want to take this linear independence to algebra character tables.

When the group algebra $k[G]$ is semisimple, we define its character table to be the character table of G. The columns are indexed by certain elements of the group, and knowing the character values at these elements is enough to give us the character values on all of G, which is a basis of $k[G]$. Importantly, these representatives themselves are not a full basis of $k[G]$.

If now $k[G]$ is replaced by any semisimple algebra A, this is our challenge. We need to find a nice set of standard elements of A that are linearly independent modulo $[A, A]$ (so that the columns of the character table are linearly independent), and such that we have an algorithm to obtain from these character values the character values on a basis of A.

It is important that our choices are natural. Since the characters of an algebra are a basis for the set of class functions of the group of units, we could for instance choose our standard elements to obtain any character table we desired. So for an algebra that is not a group algebra, we need to determine a natural set of standard elements in order for the character table to have any use at all.

We discuss the solution to the finite Hecke algebra version of this problem given in [GP00, § 8.2] in Section 4.2.

2.3 Complex Representations of Coxeter Groups

Let W be a finite group generated by reflections. It turns out that W has an associated root system, and has a presentation as a Coxeter group: for any two elements g, h of a monoid, define

$$P(g,h;a) = \underbrace{ghg\cdots}_{a \text{ total factors}},$$

so in particular, $P(g,h;2a) = (gh)^a$.

Then W is generated by a set S of involutions (called simple reflections) with the following relations:

$$s^2 = 1, \quad \forall s \in S \tag{Q}$$

$$P(s,t;m_{st}) = P(t,s;m_{st}), \quad \forall s,t \in S \tag{B}$$

for some $m_{st} \geq 2$ for all $s, t \in S$. We refer to the first set of relations as quadratic relations (Q) and the second set as braid relations (B).

Definition 2.1. *Let W be a Weyl group with simple reflections S. Then the braid monoid of W, denoted B^+, is the monoid generated by S with relations (B).*

Our main use of the braid monoid will be as a "universal" monoid for a Coxeter group and its Hecke algebra. Since both the Coxeter group algebra and the Hecke algebra are quotients of the corresponding braid monoid algebra over the same field, any relation that holds in the braid monoid holds in the associated Coxeter group and Hecke algebra.

We assume knowledge of the basic facts and classification of finite Coxeter groups (such as in [Hum90, Chapters 1-2]).

As a precursor to exploring Hecke algebra representations, we will look at some important results in the representation theory of Coxeter groups. Knowing the representations of a

Coxeter group helps us when studying the representations of the related Hecke algebra. See [Car71, §11.2].

Let V be an $\mathbb{R}$-representation of W where the action of W is generated by reflections. Let W' be a (not necessarily standard) parabolic subgroup of W, and let $V^{W'}$ be the W'-fixed subspace of V. Then we have the W'-module decomposition $V = V^{W'} \oplus V'$, where $V' = (V^{W'})^{\perp}$ has no W'-fixed vectors.

Let $\mathcal{P}_e(V')$ be the space of homogeneous polynomials on V' of degree e with action inherited from V'. Suppose U' is an absolutely irreducible submodule of $\mathcal{P}_e(V')$ with multiplicity 1 which does not occur in any $\mathcal{P}_i(V'), i < e$. U' is a subspace of $\mathcal{P}_e(V)$, so let U be the W-submodule of $\mathcal{P}_e(V)$ generated by U'.

Theorem 2.2 (Macdonald-Lusztig-Spaltenstein Induction)**.** *U is irreducible, occurs with multiplicity 1, and does not occur in $\mathcal{P}_i(V), i < e$.*

We call U the j-induced module and denote it $j_{W'}^{W}(U')$.

Definition 2.3 (Macdonald)**.** *In the setting of Theorem 2.2, let N' be the number of positive roots of W', and let $U' := \epsilon$ be the sign representation of $\mathcal{P}_e(V')$. Then we call $U := j_{W'}^{W}(U')$ a Macdonald representation.*

Let $W(\_)$ refer to the Coxeter group of a particular type.

Theorem 2.4. *Every representation of $W(A_n)$ and $W(B_n)$ is a Macdonald representation. The representations $W(A_n)$ are indexed by the partitions of size $n+1$, whereas the representations of $W(B_n)$ are indexed by double partitions of n.*

$W(D_n)$ is an index-2 subgroup of $W(B_n)$, and every irreducible representation of $W(D_n)$ is a $W(D_n)$-submodule of an irreducible $W(B_n)$-representation. When restricting from $W(B_n)$ to $W(D_n)$, the representation $\rho_{\alpha,\beta}|_{W(D_n)}$ corresponding to the double partition (α, β) is irreducible if $\alpha \neq \beta$, and splits into two irreducible representations if $\alpha = \beta$. Additionally, $\rho_{\alpha,\beta}|_{W(D_n)} = \rho_{\beta,\alpha}|_{W(D_n)}$.

MacDonald's construction in type A recovers the familiar Specht modules. If $\lambda \vdash n$, and W_{λ}^{T} is the parabolic subgroup of S_n corresponding to λ^T, then the module $V_\lambda := j_{W_\lambda^T}^{W}(\epsilon)$ is equivalent to the Specht module for S_n corresponding to λ. We parameterize these modules in terms of tableaux combinatorics.

Given $\lambda \vdash n$, a tableau t of shape λ is an injective filling of the Young diagram of λ with $1, \ldots, n$. Let $tab(\lambda)$ be the set of all tableaux of shape λ. An element $\sigma \in S_n$ acts on t by permuting its entries: if i appears in a particular box of t, then $\sigma \cdot i$ appears in the corresponding box of $\sigma \cdot t$. Let C_t, R_t be the subgroups of S_n preserving the columns (resp. rows) of t as sets.

The tabloid $\{t\}$ corresponding to t is the set of tableaux $\{\sigma \cdot t | \sigma \in R_t\}$. The polytabloid E_t corresponding to t is defined to be

$$E_t := \sum_{\sigma \in C_t} (-1)^\sigma \{\sigma \cdot t\}.$$

Theorem 2.5. *The vector space with basis $\{E_t | t \in tab(\lambda)\}$ is isomorphic to V_λ.*

The tableaux formulation leads us to the branching diagram of S_n:

Proposition 2.6.

$$V_\lambda|_{S_{n-1}} = \bigoplus_{\mu \vdash n-1, \mu < \lambda} V_\mu.$$

In addition, we have two important character formulas:

Theorem 2.7 (Frobenius' Character Formula)**.** *Let $r \geq n$ and $\nu \vdash n$, and let $s_\lambda(x_1, \dots, x_r)$ be the Schur polynomial corresponding to λ in r variables. Then*

$$\sum_{\lambda \vdash n} s_\lambda(x_1, \dots, x_r) \chi_\lambda(w_\nu) = \prod_{i \leq l(\nu)} (x_1^{\nu_i} + \cdots + x_r^{\nu_i}).$$

Given partitions $\mu \subset \lambda$ (in the usual sense), let cc_μ^λ be the number of connected components of $\lambda - \mu$, and l_μ^λ be the number of rows covered by $\lambda - \mu$ minus cc_μ^λ. We call $\lambda - \mu$ a *hook* if it doesn't contain any 2×2 blocks.

Theorem 2.8 (Murnaghan-Nakayama Rule)**.** *[GP00, Theorem 10.2.7] Let $\lambda \vdash n$, and let $w \in S_n$ be of the form $w = w' s_{n-k+1} \cdots s_{n-1}$ for some $w' \in S_{n-k}$. Then,*

$$\chi_\lambda(w) = \sum_{\mu \subset \lambda} (-1)^{l_\mu^\lambda} \chi_\mu(w'),$$

where the sum is over all partitions μ of $n - k$ such that $\lambda - \mu$ is a hook.

Remark 2.9. *Ram [Ram91] generalized Theorem 2.7 to the Hecke algebra case using the quantum Schur-Weyl duality of Section 3.3. This leads to a generalization of Theorem 2.8, which is given in Section 4.3.*

3 Hecke Algebras

We explain the importance of the finite Hecke algebra by giving (and proving the equivalence of) three definitions. The first definition is by generators and relations, the second is as the convolution algebra of Borel-biinvariant functions on a finite Chevalley group, and the third (in type A) is via a Schur-Weyl duality with the quantum group. We then prove that the three definitions are equivalent in appropriate contexts.

3.1 Generators and Relations

Let W be a finite Coxeter group with simple reflections S. For each $s \in S$, let q_s denote a parameter corresponding to s that is transcendental over $\mathbb{C}$.

Definition 3.1 (Definition 1)**.** *Given a Coxeter group W, let its generic Hecke algebra $\mathcal{H}(W)$ be a $\mathbb{C}(\{q_s, q_s^{-1}\})$-algebra with generators $T_s, s \in S$, and relations (B) from above and*

$$T_s^2 = (q_s - 1)T_s + q_s \tag{Q'}$$

Note that if s and t are conjugate in W, then we must have $q_s = q_t$. The reason for this is that since s and t are length 1, if they're conjugate in W, then T_s and T_t must be conjugate in $\mathcal{H}$ [GP00, Theorems 3.2.9b, 4.3.3]. Thus we can conjugate relation (Q') and obtain

$$T_t^2 = (q_s - 1)T_t + q_s,$$

so we have

$$q_s(T_t + 1) = T_t^2 + T_t = q_t(T_t + 1).$$

This means that $q_s - q_t = -T_t(q_s - q_t)$, so we must have $q_s = q_t$. As we will see in the next section, in applications to reductive groups we take all q_s equal to the cardinality q of a certain residue field.

Now let $w = s_{i_1} \cdots s_{i_n}$ be a reduced expression, and let $T_w = T_{s_{i_1}} \cdots T_{s_{i_n}}$. Then T_w is well defined by the braid relations: if $s_{j_1} \cdots s_{j_n}$ is another reduced expression for w, these expressions are equal in the braid monoid, and so $T_{s_{i_1}} \cdots T_{s_{i_n}} = T_{s_{j_1}} \cdots T_{s_{j_n}}$.

Proposition 3.2. *$\mathcal{H}(W)$ is spanned by $\{T_w | w \in W\}$.*

Proof. We prove that $\{T_w\}$ generates $\mathcal{H}(W)$ by showing that for all $s \in S, w \in W$,

$$T_s T_w = \begin{cases} T_{sw}, & l(sw) > l(w) \\ (q-1)T_w + qT_{sw}, & l(sw) < l(w). \end{cases}$$

The first case holds easily, since if $l(sw) > l(w)$ and $w = s_{i_1} \cdots s_{i_n}$ is a reduced expression, then $ss_{i_1} \cdots s_{i_n}$ is a reduced expression for sw. If $l(sw) < l(w)$, then we can write $w = sw'$ where $l(w) = 1 + l(w')$. Thus,

$$T_{sw} = T_s T_s T_{w'} = ((q_s - 1)T_s + q_s T_1)T_{w'} = (q_s - 1)T_w + q_s T_{w'} = (q_s - 1)T_w + q_s T_{sw}.$$

□

We now define the generic Hecke algebra $\mathcal{H}$ corresponding to $\mathcal{H}(W)$. Let $\{q_s | s \in S\}$ be transcendental parameters over $\mathbb{C}$ such that $q_s = q_t$ if s and t are conjugate, and q_s is transcendental over $\mathbb{C}(\{q_t | t \in S$ not conjugate to $s\})$. Let K be a "large enough" field: a finite Galois extension of $\mathbb{C}(\{q_s | s \in S\})$ such that $\mathcal{H}$ is split semisimple, the existence of which is guaranteed by [CR81, Proposition 7.13]. In particular, we define

$$\mathcal{H} := \mathcal{H}(W) \otimes_{\mathbb{C}(\{q_s | s \in S\})} K.$$

Let $\mathcal{A}$ be any commutative unital ring for which there exists a ring homomorphism $\theta : K \to \mathcal{A}$ sending the q_s to invertible elements of $\mathcal{A}$; we then view $\mathcal{A}$ as a K-algebra and define the specialization

$$\mathcal{H}_{A,\theta} := \mathcal{H} \otimes_K \mathcal{A},$$

where we often suppress the dependence on θ and write $\mathcal{H}_{\mathcal{A}}$ for $\mathcal{H}_{\mathcal{A},\theta}$.

The two most important specializations of $\mathcal{H}$ are $q_s \mapsto 1$ and $q_s \mapsto q$, where q is the characteristic of a finite field. Tits' Deformation Theorem will tell us that these specializations are (abstract) isomorphisms (see Section 4.1).

3.2 Convolution Algebra of B-Biinvariant Functions

Now let G be a finite Chevalley group. Let B be a Borel subgroup associated to a split maximal torus T, let $N = N_G(T)$, and let $W = N/T$. Bruhat and Tits abstracted the roles of B and N through a set of axioms. This conception leads to the theory of buildings, in which B and N can be seen as stabilizers of chambers and apartments, respectively.

In particular, from [GP00, §8.4]:

Proposition 3.3. *Let G, B, N, W be as above, and note that we can view elements of W as living in G, up to an element of $B \cap N$.*

(a) (Bruhat Decomposition)

$$G = \bigsqcup_{w \in W} BwB.$$

(b) If $w \in W, s \in S$, then $BsBwB \subset BswB \cup BwB$, and if $l(sw) = l(w) + 1$, then $BsBwB = BswB$.

See [Bum, §7] for a proof of Part (a) in type A.

Definition 3.4 (Definition 2)**.** *Let $\mathcal{H}_B$ be the $\mathbb{C}$-algebra of B-biinvariant functions $G \to \mathbb{C}$ under convolution. In other words,*

$$\mathcal{H}_B = \{\phi : G \to \mathbb{C} | \phi(bgb') = \phi(g) \text{ for all } b, b' \in B, g \in G\}$$

with multiplication

$$(\phi * \psi)(g) = \frac{1}{|B|} \sum_{x \in G} \phi(x)\psi(x^{-1}g) = \frac{1}{|B|} \sum_{x \in G} \phi(gx)\psi(x^{-1}).$$

By the (finite) Borel-Matsumoto Theorem [Bum, Theorem 2], irreducible representations of $\mathcal{H}_B$ are in bijection with irreducible representations of G with B-fixed vectors.

Now we show that this definition is the same as Definition 1, under the appropriate circumstances. Note that Definition 1 has "generic" q_s, while Definition 2 has $q_s = |BsB/B|$. If G is an untwisted Chevalley group over a finite field $\mathbb{F}_q$ (the case we're most interested in), then $q_s = q$ for all $s \in S$, and in general q_s is a power of q [Car71, §8.6, 14.1]. Here we get a sense for why Definition 1 is called the generic Hecke algebra: the Weyl group of a finite Chevalley group G is independent of the field of G. The q_s from Definition 1 are indeterminates, but in Definition 2 they are integers. So we show that Definition 1 is equivalent to Definition 2 once we have specialized the parameters q_s in Definition 1 to the values q_s in Definition 2.

Proposition 3.5. *$\mathcal{H}_B$ satisfies the defining relations (B) and (Q') of $\mathcal{H}(W)$.*

Proof. Let $\phi_w \in \mathcal{H}_B$ be the characteristic function of BwB. By the Bruhat decomposition of G, $\{\phi_w\}$ forms a basis of $\mathcal{H}(W)$, and we have the relations

$$(\phi_y * \phi_w)(g) = \frac{1}{|B|} \sum_{x \in G} \phi_y(x)\phi_w(x^{-1}g) = \frac{1}{|B|} \sum_{x \in BxB} \phi_w(x^{-1}g) = \frac{1}{|B|} |By^{-1}Bg \cap BwB|.$$

Now suppose that $y = s \in S$ such that $l(sw) = l(w) + 1$. Then $BsBg \cap BwB \subset (BsgB \cup BgB) \cap BwB$ so if we take $g \in W$ we must have $g = sw$ for this set to be nonzero, and then

$$\begin{aligned} BsBsw \cap BwB &= (BsBs)w \cap BwB \\ &\subset (B \cup BsB)w \cap BwB \\ &= (Bw \cup BsBw) \cap BwB \\ &= (Bw \cup BswB) \cap BwB \\ &= Bw. \end{aligned}$$

Therefore, $|Bw| = |B|$, so $\phi_s * \phi_w = \phi_{sw}$. By induction, this means that if $w = s_{i_1} \cdots s_{i_n}$ is a reduced expression, then $\phi_{s_{i_1}} * \cdots * \phi_{s_{i_n}} = \phi_{s_{i_1} \cdots s_{i_n}}$.

In particular, the braid relations hold in $\mathcal{H}_B$:

$$P(\phi_s, \phi_t; m_{st}) = \phi_{P(s,t;m_{st})} = \phi_{P(t,s;m_{st})} = P(\phi_t, \phi_s; m_{st}).$$

If now $y = w = s$, we set $q_s := |BsB|/|B| = |BsB/B|$. Here we have $BsBg \cap BsB \subset (BsgB \cup BgB) \cap BsB$, so g must equal either 1 or s, and we can write $\phi_s * \phi_s = \lambda\phi_1 + \mu\phi_s$. Evaluating both sides of at the identity obtains

$$\lambda = |(BsBs \cap BsB)|/|B| = |(B \cup BsB) \cap BsB|/|B| = |BsB|/|B| = q_s.$$

Now, $|BsBs/B| = |BsB/B| = q_s$, so

$$\begin{aligned}
\mu &= |(BsBs \cap BsB)/B| \\
&= |((B \cup BsB) \cap BsB)/B| \\
&= |BsBs/B| - |B/B| \\
&= q_s - 1.
\end{aligned}$$

Thus we have the quadratic relations as well: $\phi_s^2 = q_s\phi_1 + (q_s - 1)\phi_s$. □

Corollary 3.6. *Let G be a finite Chevalley group over $\mathbb{F}_q$, let W be its Weyl group, and let B be a Borel subgroup. Let $\mathcal{H}(W)$ and $\mathcal{H}_B$ be defined as above, and let $\mathcal{H}'(W)$ be the image of $\mathcal{H}(W)$ under the homomorphism $q_s \mapsto q$ (where the q_s are indeterminates). Then $\mathcal{H}'(W) \cong \mathcal{H}_B$ as $\mathbb{C}$-algebras.*

Proof. By Proposition 3.2 $\dim \mathcal{H}'(W) \leq \dim \mathcal{H}_B$, and by Proposition 3.5, $\mathcal{H}_B$ satisfies the defining relations of $\mathcal{H}'(W)$. Thus, the algebras are isomorphic. □

3.3 Centralizer of Quantum Group Action (Schur-Jimbo Duality)

Our final definition of the finite Hecke algebra is as the centralizer of a quantum group action. Quantum groups are certain deformations of universal enveloping algebras of Lie algebras, and have applications to various areas of physics. Quantum groups were studied in particular by Drinfeld [Dri85] and Jimbo [Jim85] in the 1980's, and the latter [Jim86] extended the usual Schur-Weyl duality of GL_n and S_k to the quantum group $U_q(\mathfrak{gl}_n)$ and the Type A Hecke algebra $\mathcal{H}(S_k)$. Here we leave q as a transcendental parameter to avoid issues at roots of unity.

We follow [Sun14] in this section. In type A, we can express the Hecke algebra as the centralizer of a quantum group on a tensor power of the quantum group's standard representation.

We define a quantum group to be a quasitriangular Hopf algebra. In other words, let A be a Hopf algebra over a field k with comultiplication $\Delta : A \to A \otimes A$, coidentity $\epsilon : A \to k$, and antipode $S : A \to A$. Let $\tau \in \mathrm{End}(A \otimes A)$ be the operator $x \otimes y \mapsto y \otimes x$.

Definition 3.7. *We say that A is quasitriangular if there exists a* universal R-matrix $R \in A \otimes A$ *such that*

(a) $R\Delta(x)R^{-1} = \tau(\Delta(x))$ *for all* $x \in A$,

(b) $(\Delta \otimes id)(R) = R_{13}R_{23}$ *and* $(id \otimes \Delta)(R) = R_{13}R_{12}$,

where R_{ij} refers to the image of R under the embedding $A \otimes A \hookrightarrow A \otimes A \otimes A$ into the i, j factors.

For more background on this definition, see [Rit02].

In particular, let q be an indeterminate, and let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}(q)$ with Chevalley generators $\{e_i, f_i, h_i\}, 1 \le i \le n$, and Serre relations

$$[h_i, h_j] = 0, \quad [e_i, f_j] = \delta_{ij} h_i, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j,$$

$$\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0, \qquad \mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0.$$

The values a_{ij} define a Cartan matrix. Let R be the corresponding root system, and W the Weyl group of R.

Let $U := U_q(\mathfrak{g})$ be the unital associative $\mathbb{C}(q)$-algebra with generators $\{E_i, F_i, K_i, K_i^{-1}\}$ and deformed Serre relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \quad [K_i, K_j] = 0,$$

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q^{-a_{ij}} F_j, \quad [E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}},$$

$$[E_i, E_j] = [F_i, F_j] = 0 \qquad \text{if } a_{ij} = 0,$$

$$E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 = F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 \qquad \text{if } a_{ij} = -1.$$

If $\mathfrak{g}$ is not simply laced, there is the possibility that $a_{ij} \neq 0, -1$; we ignore that case here. Note that some authors write $q^{\pm h_i}$ for $K^{\pm 1}$.

Let V be the standard representation of U in the sense of [CP94, §10.1-2]. In particular, V has the same dimension and weights as the standard representation of $\mathfrak{g}$, and is equal to that representation under the specialization $q \mapsto 1$. U acts diagonally on the tensor power representation $V^{\otimes k}$.

It can be shown that $U_q(\mathfrak{g})$ is quasitriangular; thus it is a quantum group. Now let $\mathfrak{g} := \mathfrak{gl}_n$, and in particular,

Proposition 3.8. *[Sun14, Corollary 2.4] There exists a matrix $R \in U \otimes U$ that satisfies Definition 3.7, and*

(a) R satisfies the Yang-Baxter equation: $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$, where R_{ij} is the map from $U \otimes U$ to $U^{\otimes m}, i, j \leq m$ given by $x \otimes y \mapsto 1 \otimes \cdots \otimes 1 \otimes x \otimes 1 \otimes \cdots \otimes 1 \otimes y \otimes 1 \otimes \cdots \otimes 1$, where x (resp. y) is the ith (resp. jth) factor.

(b) Let $\widehat{R} := \tau \circ R$. Then $\widehat{R}$ is an isomorphism $W_1 \otimes W_2 \cong W_2 \otimes W_1$ for any two representations W_1, W_2 of U.

(c) $\widehat{R}_{23}\widehat{R}_{12}\widehat{R}_{23} = \widehat{R}_{12}\widehat{R}_{23}\widehat{R}_{12}$

(d) R acts on the tensor square $V \otimes V$ of the standard representation as

$$R|_{V \otimes V} = q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q - q^{-1}) \sum_{i > j} E_{ij} \otimes E_{ji}.$$

Suppose $n > k$.

Definition 3.9 (Definition 3). *Define the Hecke algebra associated to $V^{\otimes k}$ to be*

$$\mathcal{H}_{U,k} := End_U(V^{\otimes k}).$$

By [BMP05], $\mathfrak{g}$ is "strongly rigid", meaning that U is abstractly isomorphic to the universal enveloping algebra of $\mathfrak{g}$. In particular, U is a semisimple algebra that has the same representation theory as $\mathfrak{g}$. By the Double Centralizer Theorem [GW09, Theorem 4.1.13], $U = \mathrm{End}_{\mathcal{H}_{U,k}}(V^{\otimes k})$.

Theorem 3.10. *$\mathcal{H}_{U,k} \cong \mathcal{H}(S_k)$, as defined in Definition 1.*

Proof. As U is a deformation of $\mathfrak{g}$, the dimension of its centralizer is the same: $k!$. Thus if we can find a copy of $\mathcal{H}(S_k)$ inside $\mathcal{H}_{U,k}$, they are isomorphic.

By Proposition 3.8(d),

$$\widehat{R}|_{V \otimes V} = q \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ji} \otimes E_{ij} + (q - q^{-1}) \sum_{i > j} E_{jj} \otimes E_{ii},$$

so under restriction to $V \otimes V$,

$$(\widehat{R} + q^{-1}) = (q+q^{-1}) \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} E_{ji} \otimes E_{ij} + q \sum_{i>j} E_{jj} \otimes E_{ii} + q^{-1} \sum_{i>j} E_{ii} \otimes E_{jj}$$

$$(\widehat{R} - q) = \sum_{i \neq j} E_{ji} \otimes E_{ij} - q^{-1} \sum_{i>j} E_{jj} \otimes E_{ii} - q \sum_{i>j} E_{ii} \otimes E_{jj},$$

so

$$(\widehat{R} + q^{-1})(\widehat{R} - q) = \sum_i E_{ii} \otimes E_{ii} - \sum_{i>j} E_{jj} \otimes E_{ii} - \sum_{i>j} E_{ii} - E_{jj} = 0.$$

If we replace q by $q^{1/2}$ and set $\sigma_i := q^{1/2} \widehat{R}_{i,i+1}$, the σ_i satisfy the quadratic relations (Q'). In addition, by Proposition 3.8(c), the σ_i satisfy the braid relations, so $\mathcal{H}' := \langle \sigma_i \rangle$ is a quotient of the Hecke algebra $\mathcal{H}(S_k)$.

Therefore, we can write $\sigma_w := \sigma_{i_1} \cdots \sigma_{i_n}$, where $w = s_{i_1} \cdots s_{i_n}$ is a reduced expression, and this is well-defined. Notice that under the specialization $q \mapsto 1$, the action of σ_i is the permutation action that switches the i-th and $(i+1)$-th factors of V. Thus all the σ_w are linearly independent, so in fact $\mathcal{H}' \cong \mathcal{H}(S_k)$ since they have the same dimension.

By taking $W_1 = W_2 = V$ in Proposition 3.8(d) and acting by U diagonally, we see that each $\sigma_i \in \mathrm{End}_U(V^{\otimes k}) = \mathcal{H}_{U,k}$. Thus, $\mathcal{H}(S_k) \subset \mathcal{H}_{U,k}$, and since they have the same dimension, they are isomorphic. □

Corollary 3.11. *As a $(\mathcal{H}(S_k) \otimes U)$-module, we can write*

$$V^{\otimes k} = \bigoplus_{\lambda \vdash k} V_\lambda \otimes L_\lambda,$$

where V_λ and L_λ are deformations of the representations of S_k and GL_n corresponding to λ. In particular, there is an equivalence of categories

$$V_\lambda \mapsto V_\lambda \otimes_{\mathcal{H}(S_k)} V^{\otimes k} = L_\lambda$$

from representations of $\mathcal{H}(S_k)$ to representations of $U_q(\mathfrak{gl}_n)$ that appear in $V^{\otimes k}$.

Proof. This follows from Theorem 3.10 and the Double Centralizer Theorem [GW09, Theorem 4.1.13]. □

Remark 3.12. *If $k \geq n$, the same construction may be defined, but the centralizer of U is a quotient $n\mathcal{H}_k(q)$ of $\mathcal{H}_{U,k}$ (see [Mar92, Theorem 1]).*

Remark 3.13. *In types B, C, and D, a similar duality exists, with the Hecke algebra replaced with the BMW algebra (see [BW89], [Hu07]).*

4 Hecke Algebra Representation Theory

4.1 Tits' Deformation Theorem

Tits' Deformation Theorem is a remarkable result that says that the generic Hecke algebra $\mathcal{H}$ is abstractly isomorphic both to its group algebra and to the Hecke algebra obtained by specializing $q_s \to q$ for most $q \in \mathbb{C}$. What this means is that we can study the representation theory of the generic Hecke algebra, and thus the all-parameters-equal Hecke algebra by studying the representation theory of the group algebra.

Let $q \in \mathbb{C}$, and consider the homomorphism $q_s \mapsto q$ for all $s \in S$. Denote the corresponding Hecke algebra by $\mathcal{H}_q$. For most values of q (everything but 0 and nontrivial roots of unity), $\mathcal{H}_q$ is split semisimple [GJ11].

Theorem 4.1 (Tits' Deformation Theorem)**.** *[GP00, Theorems 7.4.6, 8.1.7] If $\mathcal{H}_q$ is split semisimple, then $\mathcal{H} \cong \mathcal{H}_q \otimes_{\mathbb{C}} K$. In particular, if $q = 1$ $\mathcal{H}_q = \mathbb{C}[W]$, so $\mathcal{H} \cong K[W]$. Therefore, $\mathcal{H}$ and W have the same representation theory, so there exists a bijection $\chi \mapsto \chi'$ of irreducible characters of $\mathcal{H}$ and W, where the map $\chi \mapsto \chi'$ is induced by the specialization $q_s \mapsto 1$.*

The isomorphisms between $\mathcal{H}$ and $K[W]$ means that these algebras have "the same" representation theory in a certain sense. In particular these algebras have the same branching diagrams and the same character tables after specialization. Of course, this equality must respect the isomorphism $\mathcal{H} \cong K[W]$, and in the next section we give a way of defining the character table of $\mathcal{H}$, so that after specialization $q_s \mapsto 1$ we obtain the character table of W.

Tits' Deformation Theorem also justifies our flexibility to choose the parameters q_s to be generic or not, equal or unequal, depending on the needs of the context at hand.

Lusztig and others (see [Lus81], [Gec11]) have constructed the explicit isomorphism in all types. From this one can construct a character formula, but the isomorphisms are complex, and there are other ways of computing the character values. We mostly use Tits' Deformation Theorem to say that an isomorphism *exists* rather than refer to the actual map.

4.2 The Hecke Algebra Character Table

By Theorem 4.1, we can associate an irreducible character χ of $\mathcal{H}$ with an irreducible character of W, which we also call χ. This identification should not cause confusion since W and $\mathcal{H}$ have the same representation theory, and when evaluating character values, we can see whether the inputs are elements of W or $\mathcal{H}$.

Since $\mathcal{H}$ is an algebra, not every character value is given in a character table for the obvious reason that there are infinitely many of them. In the case of a group algebra, we index the columns of the character table by the conjugacy classes on the group, so every group element corresponds to a column, and therefore we know the character values on a nice basis of the algebra. If we just used Tits' Deformation Theorem directly, we would obtain character values on conjugacy classes of $\mathcal{H}^\times$ that form a basis of $\mathcal{H}$, but Tits' Deformation Theorem doesn't give us the explicit isomorphism, so this approach doesn't give us a way to actually compute characters on the elements T_w. On the other hand, character values on an element T_w are not determined by the conjugacy class of w.

We can solve this problem by proving that Hecke algebra character values are constant on elements of the form T_w where w is of minimal length in its conjugacy class (see Geck and Pfeiffer in [GP00, §8.2]; first done by Starkey in Type A). Geck and Pfeiffer show how to use these values to find the character values on every T_w. Thus, we can define the character table of $\mathcal{H}$ to be the square matrix with rows corresponding to irreducible characters and columns corresponding to conjugacy classes of W, and where the (χ, C)-entry is $\chi(w_C)$, w_C any minimal length element of C. Since characters are trace functions, we can work modulo $[\mathcal{H}, \mathcal{H}]$. Let $CC(W)$ be the set of conjugacy classes of W, and for each conjugacy class C of W, let w_C denote a minimal length element of C.

Theorem 4.2. *[GP00, Proposition 8.2.7] If $w \in W$,*

$$T_w \equiv \sum_{C \in CC(W)} f_{w,C} T_{w_C} \mod [\mathcal{H}, \mathcal{H}],$$

where the $f_{w,C} \in \mathbb{Z}[q_s | s \in S]$, and

(a) If w is minimal length in $C' \in CC(W)$, then $f_{w,C} = \delta_{CC'}$,

(b) If w is not minimal length in its conjugacy class, then there exists some w' with $T_w \equiv T_{w'} \mod [\mathcal{H}, \mathcal{H}]$, and there exists some $s \in S$ such that $l(sw's) < l(ws) < l(w')$ and

$$f_{w,C} = q_s f_{sws,C} + (q_s - 1) f_{ws,C}.$$

Using this result, we have an inductive construction of the values $\chi(T_w)$ given the character values $\chi(T_{w_C})$. Therefore, we can index the columns of our character table by the T_{w_C}. The character table of $\mathcal{H}$, so defined, is square, invertible, and under the specialization $q_s \mapsto 1$, recovers the character table of W.

4.3 Computing the Hecke Algebra Character Table via Parabolic Subgroups

Now we present a set of formulas for the character table of $\mathcal{H}$, the Murnaghan-Nakayama rules. These are recursive formulas for the characters of the classical types, the name of which is taken from the well-known S_n character formula. We build up the character table of W from the character tables of its standard parabolic subgroups.

Recall the following from Section 2.3. Given partitions $\mu \subset \lambda$, let cc_μ^λ be the number of connected components of $\lambda - \mu$, and l_μ^λ be the number of rows covered by $\lambda - \mu$ minus cc_μ^λ. We call $\lambda - \mu$ a *hook* if it doesn't contain any 2×2 blocks.

Theorem 4.3 (Murnaghan-Nakayama rule, type A)**.** *[GP00, Theorem 10.2.5] Let $\lambda \vdash n$, and let $W = S_n$. Let $w \in W$ satisfy $w = w' s_{n-k+1} s_{n-k+2} \cdots s_{n-1}$ for some $w' \in S_{n-k}$. Then*

$$\chi_\lambda(T_w) = \sum_{\mu \subset \lambda} (q-1)^{cc_\mu^\lambda - 1} (-1)^{l_\mu^\lambda} q^{k - l_\mu^\lambda - cc_\mu^\lambda} \chi_\mu(T_{w'}),$$

where the sum is over all partitions μ of $n-k$ such that $\lambda - \mu$ is a hook.

Notice that setting $q = 1$ recovers the classical Murnaghan-Nakayama rule (Theorem 2.8). We give an example computation of the $\mathcal{H}(S_3)$ character table using the Murnaghan-Nakayama rule in Appendix B.1.

Now let $W = W(B_n)$, and define the following quantities. Let $\pi = (\pi_1, \pi_2)$ be a double partition of n, and let $w_\pi \in W$ be a minimal-length element in C_π. If $\pi_1 = \emptyset$, let $\epsilon = 2$, and let $\epsilon = 1$ otherwise. Let k be the last part of π_ϵ, and let ρ be the resulting double partition of $n - k$ obtaining by removing the last part of π_ϵ.

Let λ be a double partition of n, and let DP_π^λ be the set of double partitions $\mu \subset \lambda$ where $\lambda \vdash n, \mu \vdash n-k$ such that $\lambda - \mu$ is contained in either λ_1 or λ_2, and furthermore, $\lambda - \mu$ is a horizontal strip if $\epsilon = 1$ and a hook if $\epsilon = 2$.

Let $q := q_s, Q := q_t$, where in the expression $W(B_n) = S_n \rtimes C_2$, $s \in S_n, t \in C_2$. Let d_μ^λ be the content of the box directly under $\lambda - \mu$, and let $Q_{\tau(\lambda-\mu)}$ be defined in the case where $\lambda - \mu$ is a hook to be Q if $\lambda - \mu \subset \lambda_1$, and -1 if $\lambda - \mu \subset \lambda_2$.

Then we have the following formula:

Theorem 4.4 (Murnaghan-Nakayama rule, type B)**.** *[Pfe97, Theorem 13.4]*

$$\chi_\lambda(T_{w_\pi}) = \begin{cases} \sum_{\mu \in DP_\pi^\lambda} (q-1)^{cc_\mu^\lambda - 1} (-1)^{l_\mu^\lambda} q^{k - l_\mu^\lambda - cc_\mu^\lambda} \chi_\mu(t_{w_\rho}), & \text{if } \epsilon = 1 \\ \sum_{\mu \in DP_\pi^\lambda} Q_{\tau(\lambda-\mu)} (-1)^{l_\mu^\lambda} q^{n + d_\mu^\lambda} \chi_\mu(T_{w_\rho}), & \text{if } \epsilon = 2, \end{cases}$$

Let $(\alpha, \beta) \vdash n$. Consider the character $\chi_{(\alpha,\beta)}$ of $W(B_n)$, and its restriction $\chi'_{(\alpha,\beta)}$ to $W(D_n)$. Recall the following from Theorem 2.4. The irreducible representations of $W(D_n)$ all arise from restrictions of irreducible representations of $W(B_n)$. If $\alpha \neq \beta$, then $\chi'_{(\alpha,\beta)}$ is irreducible, and $\chi'_{(\alpha,\beta)} = \chi'_{(\beta,\alpha)}$

If $\alpha = \beta$, $\chi'_{(\alpha,\alpha)}$ splits into two unequal irreducible characters, which we call $\chi_{(\alpha,+)}$ and $\chi_{(\alpha,-)}$. In particular, n is even, and $\alpha \vdash \frac{n}{2}$.

Theorem 4.5 (Murnaghan-Nakayama rule, type D)**.** *[GP00, §10.4]*

If $\alpha \neq \beta$, then

$$\chi'_{(\alpha,\beta)} = \chi'_{(\beta,\alpha)} = (\chi_{(\alpha,\beta)})_{Q\mapsto 1}.$$

Now if $\gamma \vdash n$ has all parts even, and $\gamma' \vdash \frac{n}{2}$ is the partition with $\gamma'_i = \frac{1}{2}\gamma_i$, let $T'_{\gamma'}$ be the element of $\mathcal{H}(S_{n/2})$ corresponding to w'_γ, but with parameter q^2. Then

$$\chi_{(\alpha,\pm)}(T_{w_{(\gamma,\emptyset)}}) = \frac{1}{2}\left(\chi_{\alpha,\alpha}(T_w) \pm (q+1)^m \chi_\alpha(T'_{\gamma'})\right).$$

Finally, if w is a representative not of this form, then

$$\chi_{(\alpha,\pm)}(T_w) = \frac{1}{2}\chi_{(\alpha,\alpha)}(T_w).$$

4.4 Computing the Hecke Algebra Character Table via Deformation of the Group Character Table

Although the Murnaghan-Nakayama rules give us the Hecke algebra character table, we might also look for a combinatorial formula

We can also use Tits' Deformation Theorem more directly: using the construction of Section 4.2, the Hecke algebra character table specializes to the group algebra character table. Let χ be an irreducible character of $\mathcal{H}$, and $w \in W$ be a minimal element in its conjugacy class. This gives us two ways to express the Hecke algebra character table.

Proposition 4.6. *$\chi(T_w)$ is given by polynomials in the q_s:*

$$\chi(T_w) = \sum_{i_j} a_{\chi;i_1,\dots,i_n} q_{s_i}^{i_j}, \qquad \left(\textit{where } \chi(w) = \sum_{i_j} a_{\chi;i_1,\dots,i_n}\right).$$

Or by the orthogonality of the Coxeter group character table:

Proposition 4.7.

$$\chi(T_{w_\lambda}) = \sum_{\nu\vdash n} \chi(w_\nu) p_\lambda^\nu, \qquad (p_\lambda^\nu \in K)\,.$$

Starkey's Rule takes the tactic of Proposition 4.7. It was the first Hecke algebra character table result, proven by Starkey in his 1975 PhD thesis at the University of Warwick. Starkey gave the first conception of a Hecke algebra character table, and gave a combinatorial formula for type A, using the associated character of the Weyl group.

Let (ρ, V) be the reflection representation of $W = S_n$. Let $\lambda \vdash n$, and let $w_\lambda \in W$ be a minimal-length element in the conjugacy class C_λ. In particular, due to the cycle structure of permutations, w_λ is a Coxeter element of the parabolic subgroup W_λ. Let $(\rho_\lambda, V_\lambda)$ be the reflection representation of W_λ.

See [GP00, §3.4] for a description of the labelling of the characters and conjugacy classes.

Theorem 4.8 (Starkey's Rule)**.** *[Gec99, Theorem 3.1]*

$$\chi(T_{w_\lambda}) = \sum_{\nu\vdash n} \chi(w_\nu) p_\lambda^\nu,$$

where

$$p_\lambda^\nu = \frac{|C_\nu \cap S_\lambda|}{|S_\lambda|} \det(q \cdot id_{V_\lambda} - \rho_\lambda(w_\nu)).$$

By [GP00, §7.3.11], the characters of $\mathcal{H}$ form a basis for the trace functions on $\mathcal{H}$. Therefore, the character tables of W and $\mathcal{H}$ are both invertible matrices over K of the same size, and thus Starkey's Rule is really a computation of the p_λ^ν. That these coefficients have a simple form for general n, λ, ν is not obvious from the start, and the multiplicative structure of the determinant allows us to prove the formula by induction from parabolic subgroups (see Section 5).

Starkey's thesis was unpublished, and our proof is due to Geck and Pfeiffer. This proof makes up a main part of this paper. In particular, we explore the possibility of computing a similar formula for other types. For general W not every conjugacy class contains a Coxeter element of a standard parabolic subgroup, and this ends up being a major roadblock in extending the formula.

We give an example computation of the $\mathcal{H}(S_3)$ character table using Starkey's rule in Appendix B.2.

Geck and Pfeiffer [GP00, §11.5] use the tactic of Proposition 4.7, and give a method of computing Hecke algebra character tables that is perhaps the least elegant, but has the virtue of working in all types.

Let $\mathcal{H}_v$ be the image of $\mathcal{H}$ under the specialization $q_s \mapsto v^{2c_s}$ for some values $c_s \in \mathbb{Z}_{\geq 1}$. We will consider characters of this algebra, and it's possible from this to find the character values of $\mathcal{H}$ by using different combinations of the c_s. By [GP00, Corollary 9.4.2], $\chi(T_w)$ is a polynomial in v with bounded degree (see [GP00, Corollary 9.4.2]). We write

$$\chi(T_w) = \sum_{j=1}^{l} a_{\chi,w,j} v^j.$$

Thus we have $l|CC(W)|^2$ unknown constants, so Geck and Pfeiffer set up linear equations to uniquely determine the $a_{\chi,w,j}$. These equations come from a wide range of results about Hecke algebras and their characters, such as induction from parabolic subalgebras, chains of parabolic subgroups, specialization to 1 and other roots of unity, generic degrees, and p-blocks of the Weyl group.

This procedure was used in the first computation of the character table of $\mathcal{H}(E_8)$ by Geck and Michel [GM97], [Gec+96].

5 Proof of Starkey's Rule

In this section, we go through a mostly complete proof of Starkey's rule. Particular care is paid to the generality of each result. In order to generalize this proof, we would need to apply each result to types other than type A. However, Propositions 5.8 and 5.12 in particular are bottlenecks.

5.1 The Braid Monoid

We start with a relation in the braid monoid that gives us the corresponding relation in the Hecke algebra.

Both $K[W]$ and $\mathcal{H}$ are quotients of the braid monoid algebra $K[B^+]$. On the other hand, by Matsumoto's Theorem [GP00, Theorem 1.2.2], there is a unique map of sets from W to B^+ that sends every element of S to itself and preserves the braid relations. Let $B^+(w)$ denote the element of B^+ corresponding to w under this map; we can think of $B^+(W)$ as corresponding to W. A corollary of Matsumoto's Theorem is that a reduced expression for an element $w \in W$ is also an expression for $B^+(w)$ in B^+. So when we want to find out if a relation in W holds in $\mathcal{H}$, our main technique is to check it in B^+.

Let w_c be a Coxeter element of W, and let w_0 be the longest element. Let $h = |w_c|$ be the Coxeter number of W.

Proposition 5.1. $B^+(w_c)^h = B^+(w_0)^2$.

Proof. We sketch the proof here. The full result is [GP00, Proposition 4.3.4].

First assume that $w_c = w_1 w_2$ where w_1 and w_2 are each products of simple reflections with the property that every simple reflection in w_1 (resp. w_2) commutes with every other simple reflection in w_1 (resp. w_2). W contains such an element since every Dynkin diagram of a finite Coxeter group is cycle free.

Note that w_1 and w_2 are both involutions, and therefore $\langle w_1, w_2\rangle$ is a dihedral group of order $2h$. In particular, $P(w_1, w_2; h) = P(w_2, w_1; h)$. Set w_0' to be this element; one can show by induction and looking at the sub-root-system coming from $\langle w_1, w_2\rangle$ that these expressions are reduced. Since every simple reflection of w_1 (resp. w_2) commutes with every other, we can write a reduced expression for w_0' with any $s \in S$ as its first factor. Thus, $l(sw_0') < l(w_0')$ for any $s \in S$, so $w_0' = w_0$. Since $P(w_1, w_2; h) = P(w_2, w_1; h)$ are reduced expressions, by Matsumoto's Theorem,

$$B^+(w_0)^2 = B^+(P(w_1, w_2; h)) \cdot B^+(P(w_2, w_1; h)) = (B^+(w_1)B^+(w_2))^h = B^+(w_c)^h.$$

Now suppose that w_c' is another Coxeter element. It is possible to show that w_c and w_c' are "conjugate in the braid group", which implies that $xw_c^m = w_c'^m x$ for some $x \in B^+$. One can show that $B^+(w_0)^2$ is in the center of B^+ and that B^+ has the cancellation property, so we must then have $B^+(w_0)^2 = B^+(w_c)^h$. □

5.2 The Main Argument

We follow [GP00]. The first key step is a result by Springer that gives the action of the central element $T_{w_0}^2$ in a representation affording χ. This allows us to find a formula for the character value of T_{w_c} in terms of exterior powers of the reflection representation of S_n, where w_c is a Coxeter element, and then Frobenius reciprocity takes us the rest of the way. The fact underlying this last step is that every conjugacy class of S_n contains a minimal length element that is a Coxeter element of a standard parabolic subgroup. This assertion does not hold for type B unless we expand our class of subgroups to all reflection subgroups, and this is what keeps a similar argument from working for that group.

Our result is in type A, but we will aim to keep the maximal generality for as long as possible, so for now let (W, S) be any Coxeter group, with generic Hecke algebra $\mathcal{H}$ (see Section 4.1).

Let $\chi \in \mathrm{Irr}(K\mathcal{H})$, and suppose ρ is an irreducible representation with character χ. Let S' be a set of representatives of S under W-conjugacy and N_s be the number of total times any $s' \sim s$ appears in a reduced word for w_0 (well-defined by Matsumoto's Theorem).

With these quantities defined, we can state and prove the Springer result on the action of $T_{w_0}^2$.

Proposition 5.2 (Springer). *$T_{w_0}^2$ acts by the scalar*

$$z_\chi := \prod_{s\in S'} q_s^{f_s}, \qquad f_s := N_s\left(1+\frac{\chi(s)}{\chi(1)}\right).$$

Proof. $B^+(w_0)^2$ is central in the braid monoid, so $T_{w_0}^2$ is central in $\mathcal{H}$, and thus acts by a scalar z_χ. Thus we have $\det(\rho(T_{w_0}^2)) = z_\chi^m$, where $m = \chi(1)$. Take a reduced expression $w_0 = s_{i_1}\cdots s_{i_N}$, and then

$$z_\chi^m = \det(\rho(T_{s_{i_1}}))^2 \cdots \det(\rho(T_{s_{i_N}}))^2$$

since the quadratic relation does not occur in a reduced expression.

Now, to compute $\det(\rho(T_s))$, note that the the quadratic relation tells us that $T_s^2 = q_sT_1 + (q_s-1)T_s$, so if e is an eigenvalue of $\rho(T_s)$, then $e^2 = q_s + (q_s-1)e$; in other words, $e = -1$ or q_s. We can write the multiplicity of -1 as k_s and of q_s as h_s, and then $\chi(1) = \chi(T_1) = k_s + h_s$ and $\chi(T_s) = h_sq_s - k_s$, so $\chi(s) = h_s - k_s$. Therefore, $h_s = \frac{\chi(1)+\chi(s)}{2}$ and $k_s = \frac{\chi(1)-\chi(s)}{2}$, and

$$\det(\rho(T_s)) = (-1)^{k_s} q_s^{h_s}.$$

Then we have

$$z_\chi^m = \prod_{i=1}^{n} (-1)^{2k_{s_i}} q_{s_i}^{2h_{s_i}} = \prod_{s\in S'} q_s^{N_s(\chi(1)+\chi(s))} = \prod_{s\in S'} q_s^{mN_s(1+\chi(s)/\chi(1))} = \prod_{s\in S'} q_s^{mf_s}.$$

By taking m-th roots, we have the result up to a root of unity, so $z_\chi = \zeta\prod_{s\in S'} q_s^{f_s}$ for some $\zeta\in\mathbb{C}$. Specializing $q_s \mapsto 1$, we see that $w_0^2 = 1$ acts by the scalar ζ, which must therefore equal 1. $\square$

Proposition 5.2 is our foothold, and we use it to prove the following result by Broué and Michel that gives a formula for character values at certain elements.

Proposition 5.3 (Broué-Michel). *Suppose $w \in W$ with $|w| = d$, and $T_w^d = T_{w_0}^{2r}$ for some r. Let e be the exponent of W, so that in particular e/d is an integer, and let ν_s be an e-th root of q_s in KH. Then*

$$\chi(T_w) = \chi(w) \prod_{s\in S'} \nu_s^{f_s re/d}.$$

Proof. Let $\lambda_1, \dots, \lambda_m$ be the eigenvalues of T_w in a representation afforded by χ. Then since $T_w^d = T_{w_0}^{2r}$, $\lambda_1^d, \dots, \lambda_m^d$ are the eigenvalues for $T_{w_0}^{2r}$; by Lemma 5.2, they are all equal to z_χ^r. Thus, for d-th roots of unity ζ_i,

$$\chi(T_w) = \sum_{i=1}^{m} \zeta_i \prod_{s \in S'} \nu_s^{f_s re/d},$$

and specializing $q \mapsto 1$ gives us $\chi(w) = \sum_{i=1}^m \zeta_i$, so

$$\chi(T_w) = \chi(w) \prod_{s \in S'} \nu_s^{f_s re/d}.$$

□

Starkey's Rule in the Coxeter element case comes about by interpreting the pieces of Proposition 5.3. The next three lemmas are facts we will need about Coxeter groups on the way to this result.

Lemma 5.4. $T_{w_c}^h = T_{w_0}^2$.

Proof. This follows from Proposition 5.1 since these elements are equal in the braid monoid. □

Lemma 5.5. *Let M_s be the number of elements of S conjugate to s. Then $N_s = \frac{hM_s}{2}$.*

Proof. Given two roots α and β in the root system corresponding to W, the simple reflections $s_\alpha, s_\beta \in S$ corresponding to these roots are conjugate by some element $w \in W$ if and only if w sends α to β. Let's call two roots *similar* if the reflections corresponding to them are conjugate. Thus, in a reduced expression $s_{i_1} \cdots s_{i_m}$ for some element $w \in W$, the number of s_{i_k} conjugate to some $s_\alpha \in S$ is equal to the number of positive roots similar to α that w sends to negative roots. In particular, N_{s_α} is the number of positive roots similar to α.

Recall from the proof of Proposition 5.1 that $w_0 = P(w_1, w_2; h) = P(w_2, w_1; h)$, where w_1 and w_2 are particular elements of W where $w_1 w_2 = w_c$. Since the two products are equal and both reduced, they must have the same number of simple reflections conjugate to s. So by averaging these numbers, we see that $N_s = \frac{hM_s}{2}$. □

Lemma 5.6. *Let ρ be a degree-n representation of a group G, and let $\chi^{(d)}$ be the character of the d-th exterior power of ρ. Then for all $g \in G$,*

$$\det(q - \rho(g)) = \sum_{d=0}^{n} (-1)^d \chi^{(d)}(g) q^{n-d}.$$

Proof. The left side is the characteristic polynomial of $\rho(g)$, and so the coefficient of q^{n-d} is $(-1)^d$ times the d-th elementary symmetric polynomial in the eigenvalues of $\rho(g)$. But $\chi^{(d)}(g)$ is exactly the d-th elementary symmetric polynomial in the eigenvalues of $\rho(g)$, so the expressions are equal. $\square$

Now for each $s \in S'$, let (W_s, S_s) be the parabolic subgroup of (W, S) generated by all simple reflections conjugate to s. Define the representation (ρ_s, V_s) to be the reflection representation of W_s. Except in Type F_4, we will show that we can view V_s as a representation of W by showing that there is a surjective map $W \to W_s$. Define a map $S \to S_s$ by $s' \mapsto s'$ if $s' \sim s$ and $s' \mapsto 1$ otherwise. To show this map is well-defined, we need to show that the kernel of the map doesn't intersect W_s. This follows from the next lemma.

Lemma 5.7. *Suppose W is not of Type F_4. Let $S^s = S \setminus S_s$. Then no nontrivial element of $\langle S^s \rangle$ is conjugate to any element of $\langle S_s \rangle$.*

Proof. If $|S'| = 1$, the result is trivial, so assume $|S'| > 1$. Since by assumption, $W \neq F_4$, we know that (without loss of generality), $S^s = \{t\}$. So we need only show that t is not conjugate to any expression $s_{i_1} \cdots s_{i_n}$.

If $s' \in S_s$, then $m_{s't}$ must be even; otherwise $s = P(t, s'; 2m_{s't} - 1)$, and s' must be conjugate to t since $2m_{s't} - 1 \equiv 1 \mod 4$. This means that in any expression for an element of W, the parities of the numbers of the simple reflections in S_s and T_s are invariant. In particular, for an element of $\langle S^s \rangle$ to be conjugate to something in $\langle S_s \rangle$, its length must be even, but $l(t) = 1$. $\square$

With this definition, we can give the following formula for the character values at a Coxeter element of a type A or B Coxeter group.

Proposition 5.8. *Let W be of Type A or Type B. Then*

$$\chi(w_c) = \begin{cases} (-1)^{\sum d_s}, & \chi = \bigotimes_{s \in S'} \chi_s^{(d_s)} \\ 0, & \textit{otherwise.} \end{cases}.$$

Proof. Suppose that $\chi = \bigotimes_{s \in S'} \chi_s^{(d_s)}$, and first suppose $|S'| = 1$ (so W is type A). By Lemma 5.6, $\chi(w_c) = (-1)^d$ times the $n - d$-th coefficient of $\det(q - \rho(w_c))$, where ρ is the reflection representation of W. By [GP00, §3.4.3], $\det(q - \rho(w_c)) = q^{n-1} + \cdots + q + 1$, so $\chi(w_c) = (-1)^d$. In the case $|S'| > 1$, χ is a tensor product of characters of reflection representations, so $\chi(w_c) = \prod_{s \in S'} (-1)^{d_s} = (-1)^{\sum d_s}$.

Now consider characters not of the above form. By the character table orthogonality relations, $\sum_{\chi \in Irr(W)} \chi(w_c)^2 = |C(w_c)| = h$ (by [Car72, Proposition 30]), where $C(w_c)$ is the centralizer of w_c. In type A_n, $h = n+1$ and in Type B_n, $h = 2n$. We already have this many characters with value ± 1 at w_c, so every other $\chi(w_c)$ must be 0. $\square$

Now we are ready to prove the Coxeter element case of Starkey's Rule in types A and B.

For every $s \in S'$, let (ρ_s, V_s) be as above, and let $\chi_s^{(d)}$ denote the character of the d-th exterior power of V_s.

We show that both sides of the formula are equal to:

$$\begin{cases} (-1)^{\sum d_s} \prod_{s\in S'} q_s^{M_s-d_s}, & \text{if } \chi = \bigotimes_{s\in S'} \chi_s^{(d_s)} \text{ for some } \{0 \le d_s < M_s : s \in S'\} \\ 0, & \text{else.} \end{cases}$$

Lemma 5.9. *We have the following formula:*

$$\frac{1}{|W|}\sum_{w\in W}\chi(w)\prod_{s\in S'}\det(q_s \cdot id_{V_s} - \rho_s(w))$$
$$= \begin{cases} (-1)^{\sum d_s} \prod_{s\in S'} q_s^{M_s-d_s}, & \textit{if } \chi = \bigotimes_{s\in S'} \chi_s^{(d_s)} \textit{ for some } \{0 \le d_s < M_s : s \in S'\} \\ 0, & \textit{else.} \end{cases}$$

Proof. By Lemma 5.6,

$$\det(q_s \cdot \mathrm{id}_{V_s} - \rho_s(w)) = \sum_{d=0}^{M_s} (-1)^d \chi_s^{(d)}(w) q^{n-d-1}$$

for all $s \in S'$, so

$$\frac{1}{|W|}\sum_{w\in W}\chi(w)\prod_{s\in S'}\det(q_s \cdot \mathrm{id}_{V_s} - \rho_s(w))$$
$$= \frac{1}{|W|}\sum_{w\in W}\chi(w)\prod_{s\in S'}\sum_{d_s=0}^{M_s}(-1)^{d_s}\chi_s^{(d_s)}(w)q_s^{M_s-d_s}$$
$$= \sum_{d_s=0,1,\ldots,M_s}\frac{1}{|W|}\sum_{w\in W}\chi(w)(-1)^{\sum d_s}\left(\prod_{s\in S'}\chi_s^{(d_s)}(w)q_s^{M_s-d_s}\right)$$
$$= \sum_{d_s=0,1,\ldots,M_s}(-1)^{\sum d_s}\frac{1}{|W|}\sum_{w\in W}\chi(w)(\bigotimes_{s\in S'}\chi_s^{(d_s)})(w)\left(\prod_{s\in S'}q_s^{M_s-d_s}\right)$$
$$= \begin{cases} (-1)^{\sum d_s} \prod_{s\in S'} q_s^{M_s-d_s}, & \text{if } \chi = \bigotimes_{s\in S'} \chi_s^{(d_s)} \text{ for some } \{0 \le d_s < M_s : s \in S'\} \\ 0, & \text{else.} \end{cases}$$

$\square$

Proposition 5.10 (Coxeter Case, Type A and B)**.** *Let W be of type A or B, and consider the case $W_\lambda = W$, and $w_\lambda = w_c$ is a Coxeter element of W. In this case,*

$$\chi(T_{w_c}) = \frac{1}{|W|} \sum_{w \in W} \chi(w) \prod_{s \in S'} \det(q_s \cdot id_{V_s} - \rho_s(w)).$$

Proof. Lemma 5.9 tells us that

$$\frac{1}{|W|} \sum_{w \in W} \chi(w) \prod_{s \in S'} \det(q_s \cdot \mathrm{id}_{V_s} - \rho_s(w))$$
$$= \begin{cases} (-1)^{\sum d_s} \prod_{s \in S'} q_s^{M_s - d_s}, & \text{if } \chi = \bigotimes_{s \in S'} \chi_s^{(d_s)} \text{ for some } \{0 \le d_s < M_s : s \in S'\} \\ 0, & \text{else.} \end{cases}$$

Now, by Lemmas 5.4 and 5.3,

$$\chi(T_{w_c}) = \chi(w_c) \prod_{s \in S'} \nu_s^{f_s e/h} = \chi(w_c) \prod_{s \in S'} \nu_s^{N_s(1+\chi(s)/\chi(1))e/h}.$$

By Lemma 5.8, $\chi(w_c) = 0$ unless $\chi = \bigotimes_{s \in S'} \chi_s^{(d_s)}$ for some $0 \le d_s \le M_s$, so we assume χ is such a tensor product. By [GP00, §5.2.1, Lemma 5.1.2], $\chi(s) = \left(\binom{M_s-1}{d_s} - \binom{M_s-1}{d_s-1}\right) \prod_{t \in S', t \ne s} \binom{M_t}{d_t}$ and $\chi(1) = \prod_{t \in S'} \binom{M_t}{d_t}$. Thus,

$$1 + \frac{\chi(s)}{\chi(1)} = 1 + \frac{\binom{M_s-1}{d_s} - \binom{M_s-1}{d_s-1}}{\binom{M_s}{d_s}} = \frac{2\binom{M_s-1}{d_s}}{\binom{M_s}{d_s}} = \frac{2(M_s - d_s)}{M_s}.$$

By Lemma 5.8, $\chi(w_c) = (-1)^{\sum d_s}$. By Lemma 5.5 $N_s = \frac{hM_s}{2}$. Therefore, in this case

$$\chi(T_{w_c}) = (-1)^{\sum d_s} \prod_{s \in S'} q_s^{M_s - d_s},$$

so both sides of our expression are equal to

$$\begin{cases} (-1)^{\sum d_s} \prod_{s \in S'} q_s^{M_s - d_s}, & \text{if } \chi = \bigotimes_{s \in S'} \chi_s^{(d_s)} \text{ for some } \{0 \le d_s < M_s : s \in S'\} \\ 0, & \text{else.} \end{cases}$$

□

Now that we have gotten the character formula for Coxeter elements, we use the inductive structure of our formula to stretch the formula to all Coxeter elements of standard parabolic subgroups. In Type A, this gives the entire character table.

Lemma 5.11. *Let $W = W_1 \times W_2$ be a Coxeter group with Hecke algebra $\mathcal{H}$. If $\mathcal{H}_1, \mathcal{H}_2$ are the generic Hecke algebras for W_1, W_2, then the character table for $\mathcal{H}$ is the Kronecker product of the character tables for $\mathcal{H}_1$ and $\mathcal{H}_2$.*

Proof. If ρ_1 and ρ_2 are irreducible representations of W_1 and W_2, respectively, then $\rho_1 \otimes \rho_2$ is an irreducible representation of W, and a counting argument tells us that this gives all the irreducible representations of W. Thus, since $K[W] \cong \mathcal{H}$, this construction (now taking ρ_1, ρ_2 to be representations of $\mathcal{H}_1, \mathcal{H}_2$) also gives all the irreducible representations of $\mathcal{H}$. In addition, the minimal conjugacy class representatives w of W are just products $w = w_1 w_2$ where w_1, w_2 are minimal conjugacy class representatives in W_1, W_2. We must have $l(w) = l(w_1) + l(w_2)$, so $T_w = T_{w_1} T_{w_2}$. Therefore, if $\rho = \rho_1 \otimes \rho_2, w = w_1 w_2$, then $\rho(T_w) = \rho(T_{w_1} T_{w_2}) = \rho_1(T_{w_1}) \otimes \rho_2(T_{w_2})$, and so $\chi(T_w) = \chi_1(T_{w_1})\chi_2(T_{w_2})$. □

Now let $W' = \bigoplus_{i=1}^k W_i$ be a standard parabolic subgroup of W, which each W_i is an irreducible Coxeter group. Given an irreducible character ψ of W', let ψ_i denote $\psi|_{W_i}$, thought of as a character on W'. Likewise, for $w \in W'$, let w_i denote the element of W that is the identity in W'/W_i and equal to w in W_i.

Lemma 5.12. *Given some $w \in W'$, suppose we know that for all irreducible characters ψ_i of each W_i that*

$$\psi_i(T_{w_i}) = \frac{1}{|W_i|} \sum_{x \in W_i} \psi_i(x) D_i(x)$$

for some functions $D_i : W_i \to K$ (dependent on w). Then if χ is now a character of W, we have

$$\chi(T_w) = \frac{1}{|W'|} \sum_{y \in W'} \chi(y) D(y),$$

where $D(y) := \prod_i D_i(y_i)$.

Proof. Let ψ be an irreducible character of W'. Then $\psi = \prod_{i=1}^k \psi_i$. We get

$$\prod_i \frac{1}{|W_i|} \sum_{x \in W_i} \psi(x) D_i(x) = \frac{1}{|W'|} \sum_{y \in W'} \psi(y) \prod_i D_i(y_i) = \frac{1}{|W'|} \sum_{y \in W'} \psi(y) D(y).$$

On the other hand, $\psi(T_w) = \prod_i \psi_i(T_{w_i})$ by Lemma 5.11. Thus we have

$$\psi(T_w) = \frac{1}{|W'|} \sum_{y \in W'} \psi(y) D(y).$$

Now let $m(\chi, \psi)$ be the multiplicity of ψ in $\chi|_{W'}$. Then we have

$$\begin{aligned}
\chi(T_w) &= \sum_{\psi \in \mathrm{Irr}(K\mathcal{H}')} m(\chi, \psi) \left(\frac{1}{|W'|} \sum_{y \in W'} \psi(y) D(y) \right) \\
&= \frac{1}{|W'|} \sum_{y \in W'} D(y) \left(\sum_{\psi \in \mathrm{Irr}(K\mathcal{H}')} m(\chi, \psi) \psi(y) \right) \\
&= \frac{1}{|W'|} \sum_{y \in W'} D(y) \chi(y). \qquad \square
\end{aligned}$$

Proof of Starkey's Rule. By Lemma 5.10,

$$\chi(T_{w_c}) = \frac{1}{|W|} \sum_{x \in W} \chi(x) \prod_{s \in S'} \det(q_s \cdot \mathrm{id}_{V_s} - \rho_s(x)),$$

so Lemma 5.12 tells us that for any $w \in W$ that is a product of Coxeter elements of standard parabolic subgroups ($w = w_{c_1} \cdots w_{c_n}$),

$$\begin{aligned}
\chi(T_w) &= \frac{1}{|W'|} \sum_{y \in W'} \chi(y) D(y) \\
&= \frac{1}{|W'|} \sum_{y \in W'} \chi(y) \prod_{i=1}^{n} \prod_{s \in S'} \det(q_s \cdot id_{V_s} - \rho_s(y_i)) \\
&= \frac{1}{|W'|} \sum_{y \in W'} \chi(y) \prod_{s \in S'} \prod_{i=1}^{n} \det(q_s \cdot id_{V_s} - \rho_s(y_i)) \\
&= \frac{1}{|W'|} \sum_{y \in W'} \chi(y) \prod_{s \in S'} \det(q_s \cdot id_{V_s} - \rho_s(y)). \qquad \square
\end{aligned}$$

Every conjugacy class of S_n has a minimal element of this form, so the result is proven for all conjugacy classes.

5.3 What We Know in Other Types

The proof of Starkey's Rule in the previous section gets us more than just type A: in type B it provides a formula for the character values at T_{w_λ} if w_λ is a Coxeter element of a standard parabolic subgroup of W. The reason the proof works for type A is that every conjugacy class of S_n has a minimal element of this form, but this is not the case for type B.

In types other than A or B, this proof does not work since Lemma 5.8 is no longer true. Theorem 4.5 lets us compute irreducible character values in type D from those in type B, but we won't attempt this here since our results for type B are incomplete.

In this section, we will explore the difficulties in extending Starkey's Rule to type B, and also mention what is known about the existence of such a combinatorial formula.

First, our proof of Starkey's Rule gives us the following partial result in type B:

Proposition 5.13. *Let W be a Coxeter group, let $\mathcal{H}$ be the generic Hecke algebra of W, and let χ be an irreducible character of $\mathcal{H}$. Let w_λ be a minimal length element in $C_\lambda \in CC(W)$. Then*

$$\chi(T_{w_\lambda}) = \sum_{\nu \in CC(W)} \chi(w_\nu) p_\lambda^\nu.$$

If $W = W(B_n)$ and λ contains a Coxeter element of a standard parabolic subgroup W_λ of W, then

$$p_\lambda^\nu = \frac{|C_\nu \cap W_\lambda|}{|W_\lambda|} \prod_{s \in S'} \det(q_s \cdot id_{V_s} - \rho_s(w_\nu)).$$

The conjugacy classes of $W(B_n)$ for which Proposition 5.13 applies are those indexed by the double partition $(\lambda_1, \lambda_2) \vdash n$ where λ_2 has at most one part [GP00, § 3.4]. So as n gets larger, Proposition 5.13 determines a smaller and smaller portion of the character table.

What can we say about p_λ^ν in the case that it is not known? Lemma 5.12 tells us that $p_\lambda^\nu = 0$ if there exists a standard parabolic subgroup containing an element of C_λ but not C_ν. On the other hand, computational evidence suggests that $p_\lambda^\nu \neq 0$ otherwise. In fact, the quantity $\frac{|W_\lambda|}{|C_\nu \cap W_\lambda|} p_\lambda^\nu$ is a monic polynomial in q for (at least) $W = W(B_n), n \leq 4$.

By orthogonality of group characters, we know that the specialization $(p_\lambda^\nu)_{q_s \mapsto 1} = \delta_{\lambda\nu}$. Additionally, by [GP00, Corollary 9.4.2], p_λ^ν must be a polynomial in the $\sqrt{q_s}$ with coefficients in $\mathbb{C}$, and if W is of classical type, then p_λ^ν is a polynomial in the q_s with coefficients in $\mathbb{Q}$ [GP00, §9.3.4a].

In types other than type A, not every conjugacy class of W contains a Coxeter element of a standard parabolic subgroup as a minimal-length element. This is what allowed us to use Proposition 5.3 in the previous section. Using reflection subgroups in general runs into problems because this removes our concept of a length function, and because parabolic induction no longer applies. Still, this approach has promise if these difficulties can be overcome.

Every conjugacy class that is cuspidal (not contained in a proper standard parabolic subgroup) of a Coxeter group of classical type *does* contain a minimal element that is "good"

[GP00, Definition 4.3.1], which can be seen as a generalization of a Coxeter element. However, the analogue to $T_{w_0}^2$ is no longer central, so we cannot do the same proof since the action in Proposition 5.2 is no longer scalar. It is conceivable that more careful analysis of these elements could produce a type B formula.

See Appendix A for a table of the p_λ^ν for $W(B_3)$.

5.4 Application: Ocneanu's Trace

Starkey's rule has perhaps a surprising application to Ocneanu traces, which are useful in the study of Jones' invariant of knots and links, and in the classification of von Neumann algebras, which are important spaces of bounded operators on Hilbert spaces.

A von Neumann algebras can be classified by looking at its "factors" (the building blocks of a von Neumann algebra), and an important question is to classify a given factor by its "type", and Ocneanu's trace can be used as a tool in this classification. The weights of Ocneanu's trace involve Schur functions, and the positivity properties of Schur functions lead to positivity properties of the trace, which allow us to construct subfactors.

Let M_1 be a von Neumann factor of a Hilbert space, and let M_0 be a subfactor of M_1. M_1 is said to be a Type II$_1$ factor if there exists a trace function τ on M_1, and this trace restricts to a trace on M_0, which is thus also a Type II$_1$ factor. Jones [Jon83] was able to use this factor-subfactor pair to create a nested sequence of Type II$_1$ factors $M_0 \subset M_1 \subset M_2 \subset \cdots$, where M_n is generated as an algebra by M_0 and a particular algebra J_n. J_n is generated by projections, called "conditional expectations" from $M_{n+1} \to M_n$, and can also be viewed as a quotient of the Hecke algebra $\mathcal{H}_n$ of type A_n (and therefore of the corresponding Artin braid group). The parameter q that appears in $\mathcal{H}_n$ and J_n, is related to the index of M_n in M_{n+1} (which is independent of n).

The trace on M_n is the same as the trace on M_{n+1} under restriction, so we may view τ as a trace on $M_\infty = \bigcup M_n$. When we restrict τ to J_n, the result is a Markov trace: whenever $h \in \mathcal{H}_{n-1}$, $\tau(hT_{s_n}) = z(q)\tau(h)$, where $z(q)$ is a function only of q.

Jones used this construction to define the Jones polynomial, which is a scalar multiple of $\tau(h)$ where h is the image of a braid in J_n. This polynomial, when seen as a function on braids, is an invariant of the associated knot/link (see [Jon83] or [Bir97]).

Ocneanu and others independently [Fre+85] extended Jones' construction to the two variable case. The resulting polynomial is called the HOMFLYPT polynomial, and it specializes to both the Jones polynomial, and the Alexander-Conway polynomial [Fre+85, p. 240]. The construction of the HOMFLYPT polynomial replaces J_n with $\mathcal{H}_n$, and the resulting trace is called an Ocneanu trace.

Now take $\mathcal{H} := \mathcal{H}_{n-1}$. A trace function τ on $\mathcal{H}$ is an Ocneanu trace with parameter $z \in K$ if $\tau(T_1) = 1$ and $\tau(T_w T_{s_m}) = z\tau(T_w)$ for any $s_m \in S$ and $W = \langle s_1, \ldots, s_{m-1} \rangle$. For every $z \in K$, there exists a unique Ocneanu trace with parameter z, which we call τ_z.

To see this, take w_μ to be the element $s_1 s_2 \cdots s_{\mu_1 - 1} s_{\mu_1 + 1} \cdots s_{\mu_1 + \mu_2 - 1} s_{\mu_1 + \mu_2 + 1} \cdots$; w_μ is a Coxeter element of S_μ and a minimal element of C_μ. From this, we have that $\tau_z(T_{w_\mu}) = z^{l(w_\mu)}$, and since these elements form a complete set of minimal representatives of the conjugacy classes of $\mathcal{H}$, this determines τ_z, and every Ocneanu trace has this form.

The irreducible characters χ_λ of $\mathcal{H}$ are a basis of the space of trace functions on $\mathcal{H}$ (since this is also true for $K[W] \cong \mathcal{H}$), and so we can write

$$\tau_z = \sum_{\lambda \vdash n} w_\lambda(z) \chi_\lambda$$

for some structural constants $w_\lambda(z) \in K$. We call these constants the weights of τ_z.

As an application of Starkey's Rule, we can determine these weights, by the argument of Geck and Jacon [GJ03].

Given $\lambda \vdash n$, let x be a box in the diagram of λ and let $c(x)$ be its content, $h(x)$ to be its hook length, and $n(\lambda) = \sum_i (i-1)\lambda_i$. We have the following formula for the weights of τ_z:

Theorem 5.14.

$$w_\lambda(z) = q^{n(\lambda)} \prod_{x \in \lambda} \frac{q - 1 + z(q^{c(x)} - 1)}{q^{h(x)} - 1}$$

We will need the following lemma (see [GJ03, p. 4]):

Lemma 5.15.

$$q^{n(\lambda)} \prod_{x \in \lambda} \frac{q - 1 + (q^{c(x)} - 1)\frac{q^r(1-q)}{1-q^r}}{q^{h(x)} - 1} = \left(\frac{1-q}{1-q^r} \right)^n s_\lambda(1, q, \ldots, q^{r-1})$$

Proof of Theorem 5.14. We must show that

$$z^{l(w_\mu)} = \sum_{\lambda \vdash n} \left(q^{n(\lambda)} \prod_{x \in \lambda} \frac{q - 1 + z(q^{c(x)} - 1)}{q^{h(x)} - 1} \right) \chi_\lambda(T_{w_\mu})$$

for all $z \in K, \mu \vdash n$. We can consider both sides of the equation to be polynomials in z, so the equation defines a one-dimensional algebraic variety, and so it is sufficient to show the equality for the infinitely-many elements

$$z_r := q^r \frac{1-q}{1-q^r}, r \in \mathbb{N}, r \geq n.$$

By Lemma 5.15, we need to show

$$q^{rl(w_\mu)} \left(\frac{1-q^r}{1-q}\right)^{n-l(w_\mu)} = \sum_{\lambda \vdash n} s_\lambda(1, q, \ldots, q^{r-1}) \chi_\lambda(T_{w_\mu}).$$

Note that $l(\mu) = n - l(w_\mu)$ since w_μ is a Coxeter element of a parabolic subgroup of S_n with $n - l(w_\mu)$ parts. We have

$$\begin{aligned}
\sum_{\lambda \vdash n} s_\lambda(1, q, \ldots, q^{r-1}) \chi_\lambda(T_{w_\mu}) &= \sum_{\lambda \vdash n} s_\lambda(1, q, \ldots, q^{r-1}) \sum_{\nu \vdash n} p_\mu^\nu \chi_\lambda(w_\nu) && \text{(Starkey's Rule)} \\
&= \sum_{\nu \vdash n} p_\mu^\nu \prod_{i \leq l(\nu)} \left(1 + q^{\nu_i} + \cdots + q^{(r-1)\nu_i}\right) && \text{(Lemma 2.7)} \\
&= \sum_{\nu \vdash n} p_\mu^\nu \prod_{i \leq l(\nu)} \frac{q^{r\nu_i} - 1}{q^{\nu_i} - 1}.
\end{aligned}$$

We can write the formula for p_μ^ν in Starkey's Rule as

$$p_\mu^\nu(q) = \frac{|C_\nu \cap S_\mu|}{|S_\mu|} (q-1)^{-l(\mu)} \prod_{i \leq l(\nu)} (q^{\nu_i} - 1)$$

(see [GP00, §3.4.3, Exercise 3.11]), so

$$\begin{aligned}
\sum_{\nu \vdash n} p_\mu^\nu(q) \prod_{i \leq l(\nu)} \frac{q^{r\nu_i} - 1}{q^{\nu_i} - 1} &= \frac{|C_\nu \cap S_\mu|}{|S_\mu|} \sum_{\nu \vdash n} (q-1)^{-l(\mu)} \left(\prod_{i \leq l(\nu)} (q^{\nu_i} - 1) \right) \left(\prod_{i \leq l(\nu)} \frac{q^{r\nu_i} - 1}{q^{\nu_i} - 1} \right) \\
&= \frac{|C_\nu \cap S_\mu|}{|S_\mu|} (q-1)^{-l(\mu)} \sum_{\nu \vdash n} \prod_{i \leq l(\nu)} (q^{r\nu_i} - 1) \\
&= \left(\frac{1-q^r}{1-q}\right)^{l(\mu)} \sum_{\nu \vdash n} p_\mu^\nu(q_r)
\end{aligned}$$

Therefore, to conclude we just need to show that

$$q^{rl(w_\mu)} = \sum_{\nu \vdash n} p_\mu^\nu(q^r).$$

But this is just the assertion of Starkey's Rule on the "trivial representation" of $\mathcal{H}$ (which sends $T_w \mapsto q^{l(w)}$), but replacing q by q^r.

$\square$

This weight computation was done first by Ocneanu (unpublished; see [Wen88] and [Jon87, §4]). Ocneanu traces have also been classified in type B and type D [GL04]. Those weights were computed by Orellana [Ore98]. One reason a generalized Starkey's rule would be applicable is it might give us a nice proof of these weights which would be computationally simpler.

6 Future Research Directions

6.1 Starkey's Rule

In Section 5, we showed that an analogous formula to Starkey's Rule exists for certain columns of the character table in type B. Namely, if W is a type B Coxeter group and χ is an irreducible character of $\mathcal{H}(W)$, then Proposition 5.13 is a new result giving us a formula for $\chi(T_w)$ for any $w \in W$ that is a Coxeter element of a standard parabolic subgroup of W.

I am working to further extend Starkey's rule to the full type B Hecke algebra character table. To be more precise, by Proposition 4.7, determining the p_λ^μ is equivalent to determining the character table. So my broad goal is to give a closed-form expression for the p_λ^μ, but I am particularly looking for a "Starkey-like" formula in the sense that it has the approximate form

$$p_\lambda^\mu = \frac{|C_\mu \cap W(\lambda)|}{|W(\lambda)|} d(w_\mu),$$

where $W(\lambda)$ is some reflection subgroup of W depending on λ, and $d(w_\mu)$ is a monic polynomial in the q_s whose coefficients involve the representations of $W(\lambda)$ evaluated at w_μ, and that is multiplicative over direct products. In Starkey's rule, this polynomial is the characteristic polynomial of the reflection representation of S_λ evaluated at w_μ, so we may hope to involve reflection representations in a generalized formula too. The desired multiplicative structure for p_λ^μ would allow us to compute character values on a subset of elements (perhaps via Proposition 5.3), and then use Lemma 5.12 to prove the formula for the whole group.

The reason the type A proof of Starkey's rule does not give an analogous formula for type B is that in $W(B_n)$ not every conjugacy class contains a minimal-length element that is a Coxeter element of a standard parabolic subgroup. Coxeter elements formed the "base" of induction in our proof of Starkey's Rule in Section 5. There are two main ongoing promising directions to make progress towards a solution.

First, one can loosen our class of elements from Coxeter elements to a larger set of elements of $W(B_n)$ that intersects every cuspidal conjugacy class. If we can prove a nice enough formula for these elements, Lemma 5.12 will give us the character table for the whole group.

Possible candidates for these elements are "good elements" as defined by Geck and Pfeiffer (see [GP00, §4.3]) or "quasi-Coxeter elements" as defined by Gobet (see [Gob16]).

The generator T_{w_0} corresponding to the longest element of W (which is alone in its conjugacy class) also satisfies Lemma 5.3, and so

$$\chi(T_{w_0}) = \chi(w_0) \prod_{s \in S'} \nu_s^{nM_s(1+\chi(s)/\chi(1))e/2}.$$

If $\chi = \bigotimes_{s \in S'} \chi_s^{(d_s)}$, computations similar to those in Proposition 5.10 give us the formula:

$$\chi(T_{w_0}) = \prod_{t \in S'} \binom{M_t}{d_t} \cdot (-1)^{\sum d_s} \prod_{s \in S'} q_s^{n(M_s - d_s)},$$

which is similar to the formula given for $\chi(T_{w_c})$ near the end of the proof of Proposition 5.10. So this suggests that there may be a "Starkey-like" formula for this element too (and therefore for longest elements of parabolic subgroups).

Second, we may expand the set of subgroups we consider beyond just the standard parabolic subgroups. Every conjugacy class of $W(B_n)$ *does* contain a minimal length Coxeter element of a reflection subgroup isomorphic to $W(A_n)$ or $W(B_n)$ [GP00, Proposition 3.4.7]. These subgroups are not necessarily parabolic (conjugate to a standard parabolic subgroup). So we can calculate the character values on these elements within those subgroups, but our induction procedure from Lemma 5.12 must be expanded to work with these more general subgroups.

6.2 Hecke Algebras of Finite Renner Monoids

This work started in the UMN Algebra and Combinatorics REU this past summer, in a project designed by Ben Brubaker. Solomon [Sol02] reduced the computation of the character table of a Renner monoid to the computation to either one of two matrices, which he labelled 'A' and 'B'. The B-matrix encodes restriction multiplicities of irreducible representations of the Renner monoid. In particular, if R is a Renner monoid of type B_n, let ρ be an irreducible representation of R and let σ be an irreducible representation of one of the subgroups $W' = W(B_n), S_k, 1 \leq k \leq n$. Then the (ρ, σ) entry in the B-matrix is the multiplicity of σ in the restriction of ρ to W'.

Solomon computed the A- and B-matrices in type A, and the four undergrads in our project computed them in type B. We were then able to use the B-matrix of type B to compute certain character values of the associated Hecke algebra.

I am working to give a rigorous definition of the character table for a general Renner monoid Hecke algebra, in the sense of Section 4.2, which would prove that the character values we computed are in fact the character table of this Hecke algebra. We are hopeful that such a technique will work. In Type A, Dieng, Halverson, and Poladian defined and computed the type A Renner monoid Hecke algebra character table [DHP03] using an explicit definition of "standard elements" on which to evaluate the characters. Our character table exactly matches theirs, which means that our conjecture gives correct standard elements in type A. The technique of Geck and Pfeiffer described in Section 4.2 is type independent, so we expect this technique will also define a type-independent Hecke algebra character table in the Renner monoid case.

This definition would mean that our technique using B-matrices gives a simple type-independent procedure to calculate Renner monoid Hecke algebra character tables of all types, given the data of character tables of the associated Renner monoids and group Hecke algebras, all of which are known (see [LLC09], [GM97]). For precise statements, see [Har+].

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References

[Bir97] Birman, Joan S. "The Work of Vaughan F. R. Jones". In: *Fields Medallist's Lectures* (1997), pp. 435–445.

[BMP05] Bordemann, Martin, Makhlouf, Abdenacer, and Petit, Toukaiddine. "Deformation Par Quantification et Rigidite des Algebres Enveloppantes". In: *Journal of Algebra* 285 (2005), pp. 623–648.

[Bum] Bump, Daniel. "Hecke Algebras". Unpublished notes. URL: `http://sporadic.stanford.edu/bump/math263/hecke.pdf`.

[BW89] Birman, Joan S. and Wenzl, Hans. "Braids, Link Polynomials, and a New Algebra". In: *Transactions of the American Mathematical Society* 313 (1989), pp. 249–273.

[Car71] Carter, Roger W. *Simple Groups of Lie Type*. John Wiley and Sons, 1971.

[Car72] Carter, Roger W. "Conjugacy Classes in the Weyl Group". In: *Compositio Mathematica* 25 (1972), pp. 1–59.

[CP94] Chari, Vyjayanthi and Pressley, Andrew. *A Guide to Quantum Groups.* Cambridge University Press, 1994.

[CR81] Curtis, Charles W. and Reiner, Irving. *Methods of Representation Theory.* John Wiley and Sons, 1981.

[DHP03] Dieng, Momar, Halverson, Tom, and Poladian, Vahe. "Character Formulas for q-Rook Monoid Algebras". In: *Journal of Algebraic Combinatorics* (2003), pp. 99–123.

[Dri85] Drinfeld, Vladimir. G. "Hopf algebras and the quantum Yang-Baxter equation". In: *Sov. Math. Dokl.* 32 (1985). [Dokl. Akad. Nauk Ser. Fiz.283,1060(1985)], pp. 254–258.

[Fre+85] Freyd, Peter J. et al. "A New Polynomial Invariant of Knots and Links". In: *Bulletin of the American Mathematical Society* 12 (1985).

[Gec+96] Geck, Meinolf Josef et al. "CHEVIE–A System for Computing and Processing Generic Character Tables". In: *Applicable Algebra in Engineering, Communication, and Computing* 7 (1996), pp. 175–210.

[Gec11] Geck, Meinolf Joseff. "On Iwahori-Hecke Algebras with Unequal Parameters and Lusztig's Isomorphism Theorem". In: *Pure and Applied Mathematics Quarterly* 7 (July 2011). DOI: 10.4310/PAMQ.2011.v7.n3.a5.

[Gec99] Geck, Meinolf Josef. "The character table of the Iwahori-Hecke algebra of the symmetric group: Starkey's Rule". In: *C. R. Acad Sci. Paris Ser. I Math* 329.5 (1999), pp. 361–366.

[GJ03] Geck, Meinolf Josef and Jacon, Nicolas. "Ocneanu's trace and Starkey's rule". English. In: *Journal of Knot Theory and its Ramifications* 12.7 (2003), pp. 899–904. ISSN: 0218-2165. DOI: 10.1142/S0218216503002834.

[GJ11] Geck, Meinolf Josef and Jacon, Nicolas. *Representations of Hecke Algebras at Roots of Unity.* Springer-Verlag London, 2011.

[GL04] Geck, Meinolf Josef and Lambropoulou, Sofia. "Markov traces and knot invariants related to Iwahori-Hecke algebras of type B". In: *Journal fur die Reine und Angewandte Mathematik* 1997 (June 2004). DOI: 10.1515/crll.1997.482.191.

[GM97] Geck, Meinolf Josef and Michel, Jean. "'Good' Elements in Finite Coxeter Groups and Representations of Iwahori-Hecke Algebras". In: *Procedings of the London Mathematical Society* 74 (1997), pp. 275–305.

[Gob16] Gobet, Thomas. "On cycle decompositions in Coxeter groups". In: *Séminaire Lotharingien de Combinatoire* (Nov. 2016).

[GP00] Geck, Meinolf Josef and Pfeiffer, Götz. *Characters of Finite Coxeter Groups and Iwahori-Hecke Algebras.* Oxford, UK: Clarendon Press, 2000.

[GW09] Goodman, Roe and Wallach, Nolan R. *Symmetry, Representations, and Invariants.* Springer Dordrecht Heidelberg, 2009.

[Har+] Hardt, Andy et al. "Characters of Renner Monoids and Their Hecke Algebras". Preprint. URL: https://arxiv.org/pdf/1811.12343.pdf.

[Hu07] Hu, Jun. "BMW Algebra, Quantized Coordinate Algebra, and Type C Schur-Weyl Duality". In: *Representation Theory* (2007).

[Hum90] Humphreys, James E. *Reflection Groups and Coxeter Groups*. Cambridge University Press, 1990.

[IM65] Iwahori, Nagayoshi and Matsumoto, Hideya. "On some Bruhat decomposition and the structure of the Hecke rings of p-adic Chevalley groups". In: *Publications Mathematiques de l'IHES* 25 (1965), pp. 5–48. URL: http://www.numdam.org/item/PMIHES_1965__25__5_0.

[Iwa64] Iwahori, Nagayoshi. "On the Structure a a Hecke Ring of a Chevalley Group over a Finite Field". In: *Journal of the Faculty of Science* 10 (1964), pp. 215–236.

[Jim85] Jimbo, Michio. "A q-Difference Analogue of $U(\mathfrak{g})$ and the Yang-Baxter Equation". In: *Letters in Mathematical Physics* 10 (1985), pp. 63–69.

[Jim86] Jimbo, Michio. "A q-Analogue of $U(gl(N+1))$, Hecke Algebra, and the Yang-Baxter Equation". In: *Letters in Mathematical Physics* 11 (1986).

[Jon83] Jones, Vaughan. "Index for Subfactors". In: *Inventiones Mathematicae* (1983), pp. 1–25.

[Jon87] Jones, Vaughan. "Hecke Algebra Representations of Braid Groups and Link Polynomials". In: *Annals of Mathematics* (1987), pp. 335–388.

[LLC09] Li, Zhuo, Li, Zhenheng, and Cao, You'an. "Representations of the Renner Monoid". In: *International Journal of Algebra and Computation* 19 (2009), pp. 511–525.

[Lus81] Lusztig, George. "On a Theorem of Benson and Curtis". In: *Journal of Algebra* 71 (1981).

[Mar92] Martin, Paul Purdon. "On Schur-Weyl Duality, A_n Hecke Algebras, and Quantum $sl(N)$ on $\otimes^{n+1}\mathbb{C}^N$". In: *International Journal of Modern Physics* 7 (1992), pp. 645–673.

[Mat64] Matsumoto, Hideya. "Generateurs et Relations des Groupes de Weyl Generalises". In: *Comptes Rendus Mathematique Academie des Sciences* 258 (1964), pp. 3419–3422.

[Ore98] Orellana, Rosa. "Weights of Markov Traces on Hecke algebras". In: *Journal fur die Reine und Angewandte Mathematik (Crelles Journal)* 1999 (July 1998). DOI: 10.1515/crll.1999.023.

[Pfe97] Pfeiffer, Götz. "Character values of Iwahori-Hecke algebras of type B". In: *Progress in Mathematics* 141 (1997).

[Ram91] Ram, Arun. "A Frobenius Formula for the Characters of the Hecke Algebras". In: *Inventiones Mathematicae* (1991), pp. 461–488.

[Rit02] Ritter, William Gordon. "Introduction to Quantum Group Theory". In: *arXiv Mathematics e-prints*, math/0201080 (Jan. 2002), math/0201080. arXiv: `math/0201080 [math.QA]`.

[Sol02] Solomon, Louis. "Representations of the Rook Monoid". In: *Journal of Algebra* (2002), pp. 309–342.

[Sun14] Sun, Yi. *Schur-Weyl Duality for Quantum Groups.* MIT-Northeastern Fall 2014 Graduate Seminar. 2014.

[Web16] Webb, Peter. *A Course in Finite Group Representation Theory.* Cambridge University Press, 2016.

[Wen88] Wenzl, Hans. "Hecke Algbras of Type A_n and Subfactors". In: *Inventiones Mathematicae* 92 (1988), pp. 349–383.

Appendix A Values of p_λ^ν in Type B_3

Let $W = W(B_3)$, and let $q := q_s, Q := q_t$, where in the expression $W = S_3 \rtimes C_2$, $s \in S_3, t \in C_2$. Below we exhibit the p_λ^ν for W. More precisely, if λ, ν are double partitions of 3, the entry in row λ and column ν is $\frac{|W_\lambda|}{|C_\nu \cap W_\lambda|} p_\lambda^\nu$. These polynomials were reverse engineered from the character table given in [Pfe97].

$$
\begin{array}{c|c}
\nu\backslash\lambda & \begin{array}{ccc} (1^3,\emptyset) & (1^2,1) & (1,1^2) \end{array} \\
\begin{array}{c} (1^3,\emptyset) \\ (1^2,1) \\ (1,1^2) \\ (\emptyset,1^3) \\ (21,\emptyset) \\ (1,2) \\ (2,1) \\ (\emptyset,21) \\ (3,\emptyset) \\ (\emptyset,3) \end{array} &
\begin{bmatrix}
1 & Q-1 & q^2Q^2+q^2-4qQ+Q^2+1 \\
0 & Q+1 & q^2Q^2-q^2+Q^2-1 \\
0 & 0 & q^2Q^2+q^2+4qQ+Q^2+1 \\
0 & 0 & 0 \\
0 & 0 & q^2Q^2+q^2-Q^2-1 \\
0 & 0 & q^2Q^2-q^2-Q^2+1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{cc}
\nu\backslash\lambda & \begin{matrix} (\emptyset,1^3) & \qquad (21,\emptyset) \end{matrix} \\
\begin{matrix} (1^3,\emptyset) \\ (1^2,1) \\ (1,1^2) \\ (\emptyset,1^3) \\ (21,\emptyset) \\ (1,2) \\ (2,1) \\ (\emptyset,21) \\ (3,\emptyset) \\ (\emptyset,3) \end{matrix} &
\begin{bmatrix}
q^6Q^3 - q^6 - 9q^4Q^2 + 4q^3Q^3 + 9q^4Q - 9q^2Q^2 & q-1 \\
q^6Q^3 + q^6 - 3q^4Q^2 + 4q^3Q^3 - 3q^4Q - 3q^2Q^2 + 4q^3 - 3q^2Q + Q^3 + 1 & 0 \\
q^6Q^3 - q^6 + 3q^4Q^2 + 4q^3Q^3 - 3q^4Q + 3q^2Q^2 - 4q^3 - 3q^2Q + Q^3 - 1 & 0 \\
q^6Q^3 + q^6 + 9q^4Q^2 + 4q^3Q^3 + 9q^4Q + 9q^2Q^2 + 4q^3 + 9q^2Q + Q^3 + 1 & 0 \\
q^6Q^3 - q^6 - 3q^4Q^2 + 3q^4Q + 3q^2Q^2 - 3q^2Q - Q^3 + 1 & q+1 \\
q^6Q^3 + q^6 - 3q^4Q^2 - 3q^4Q + 3q^2Q^2 + 3q^2Q - Q^3 - 1 & 0 \\
q^6Q^3 + q^6 + 3q^4Q^2 + 3q^4Q - 3q^2Q^2 - 3q^2Q - Q^3 - 1 & 0 \\
q^6Q^3 - q^6 + 3q^4Q^2 - 3q^4Q - 3q^2Q^2 + 3q^2Q - Q^3 + 1 & 0 \\
q^6Q^3 - q^6 - 2q^3Q^3 + 2q^3 + Q^3 - 1 & 0 \\
q^6Q^3 + q^6 - 2q^3Q^3 - 2q^3 + Q^3 + 1 & 0
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{cc}
\nu\backslash\lambda & \begin{matrix} (1,2) & \qquad (2,1) & \qquad\qquad (\emptyset,21) \end{matrix} \\
\begin{matrix} (1^3,\emptyset) \\ (1^2,1) \\ (1,1^2) \\ (\emptyset,1^3) \\ (21,\emptyset) \\ (1,2) \\ (2,1) \\ (\emptyset,21) \\ (3,\emptyset) \\ (\emptyset,3) \end{matrix} &
\begin{bmatrix}
qQ - q + Q + 1 & qQ - q - Q + 1 & q^3Q^2 + q^3 - 6q^2Q + 6qQ - Q^2 - 1 \\
qQ + q - Q - 1 & qQ + q - Q - 1 & q^3Q^2 - q^3 - Q^2 + 1 \\
qQ - q - Q + 1 & 0 & q^3Q^2 + q^3 + 2q^2Q - 2qQ - Q^2 - 1 \\
0 & 0 & q^3Q^2 - q^3 - Q^2 + 1 \\
qQ - q + Q - 1 & qQ - q + Q - 1 & q^3Q^2 + q^3 - 2q^2Q - 2qQ + Q^2 + 1 \\
qQ + q + Q + 1 & 0 & q^3Q^2 - q^3 + Q^2 - 1 \\
0 & qQ + q + Q + 1 & q^3Q^2 - q^3 + Q^2 - 1 \\
0 & 0 & q^3Q^2 + q^3 + 2q^2Q + 2qQ + Q^2 + 1 \\
0 & 0 & q^3Q^2 + q^3 - Q^2 - 1 \\
0 & 0 & q^3Q^2 - q^3 - Q^2 + 1
\end{bmatrix}
\end{array}
$$

$$
\begin{array}{cc}
\nu\backslash\lambda & \begin{matrix} (3,\emptyset) & \qquad\qquad (\emptyset,3) \end{matrix} \\
\begin{matrix} (1^3,\emptyset) \\ (1^2,1) \\ (1,1^2) \\ (\emptyset,1^3) \\ (21,\emptyset) \\ (1,2) \\ (2,1) \\ (\emptyset,21) \\ (3,\emptyset) \\ (\emptyset,3) \end{matrix} &
\begin{bmatrix}
q^2 - 2q + 1 & q^2Q - q^2 - 2qQ + 2q + Q - 1 \\
0 & q^2Q + q^2 - 2qQ - 2q + Q + 1 \\
0 & q^2Q - q^2 - 2qQ + 2q + Q - 1 \\
0 & q^2Q - 2qQ - 2Q + Q + 1 \\
q^2 - 1 & q^2Q - q^2 - Q + 1 \\
0 & q^2Q + q^2 - Q - 1 \\
0 & q^2Q + q^2 - Q - 1 \\
0 & q^2Q - q^2 - Q + 1 \\
q^2 + q + 1 & q^2Q - q^2 + qQ - q + Q - 1 \\
0 & q^2Q + q^2 + qQ + q + Q + 1
\end{bmatrix}
\end{array}
$$

Note that Proposition 5.13 gives us the (correct) values for p_λ^ν in seven of the ten columns; the exceptions are $\nu = (1,1^2), (\emptyset,1^3), (\emptyset,21)$. These hold-outs seem to have a more complicated structure than the columns we do understand, particularly the latter two, which correspond to cuspidal conjugacy classes.

Appendix B Example Computations of the $\mathcal{H}(S_3)$ character table

B.1 Murnaghan-Nakayama Rule

Recall from Section 4.3 the Murnaghan-Nakayama rule for the type A Hecke algebra:

If $\lambda \vdash n$, and $w \in S_n$ satisfy $w = w's_{n-k+1}s_{n-k+2}\cdots s_{n-1}$ for some $w' \in S_{n-k}$, then

$$\chi_\lambda(T_w) = \sum_{\mu \subset \lambda} (q-1)^{cc_\mu^\lambda - 1}(-1)^{l_\mu^\lambda} q^{k-l_\mu^\lambda - cc_\mu^\lambda} \chi_\mu(T_{w'}),$$

where the sum is over all partitions μ of $n-k$ such that $\lambda - \mu$ is a hook.

We use this rule to compute the character table for $\mathcal{H}(S_3)$. $\mathcal{S}_{\prime}$ and $\mathcal{S}_{\infty}$ are one-dimensional algebras, and the trivial representation is the only representation. Thus, the character table is made up of the single character value $\chi_{triv}(T_1) = 1$. For S_2, to save space let us take for granted the character table which can also be calculated by the Murnaghan-Nakayama rule:

$$\begin{array}{c c}
\mathcal{H}(S_2) & \begin{array}{cc} w_{(1^2)} & w_{(2)} \end{array} \\
\begin{array}{c} \chi_{(2)} \\ \chi_{(1^2)} \end{array} & \left[\begin{array}{cc} 1 & q \\ 1 & -1 \end{array}\right].
\end{array}$$

Let's start with the character value $\chi_{triv}(T_1)$. Here, $\lambda = (3)$, $w = 1, k = 1, w' = 1 \in S_2$, and the only possible μ is (2). Thus, $cc_\mu^\lambda = 1, l_\mu^\lambda = 0$, and

$$\chi_{triv}(T_1) = (q-1)^0 \cdot (-1)^0 \cdot q^0 \cdot \chi_{(2)}(T_1) = 1.$$

Moving along the row, let's calculate $\chi_{triv}(T_{s_1})$. Here, $\lambda = (3), w = s_1, k = 1, w' = s_1 \in S_2$, the only choice for μ is (2), making $cc_\mu^\lambda = 1, l_\mu^\lambda = 0$, and

$$\chi_{triv}(T_{s_1}) = (q-1)^0 \cdot (-1)^0 \cdot q^0 \cdot \chi_{(2)}(T_{s_1}) = q.$$

Now for $\chi_{triv}(T_{s_1s_2})$. Here, $\lambda = (3), w = s_1s_2, k = 3, w' = 1 \in S_0$. We must have $\mu = (0)$, so $cc_\mu^\lambda = 1, l_\mu^\lambda = 3$, and

$$\chi_{triv}(T_{s_1s_2}) = (q-1)^0 \cdot (-1)^0 \cdot q^2 \cdot \chi_{(0)}(T_1) = q^2.$$

On to the sign character: here $\lambda = (1^3)$. If $w = 1$, then $k = 1, w' = 1 \in S_2, \mu = (1^2), cc_\mu^\lambda = 1, l_\mu^\lambda = 0$, and

$$\chi_{sign}(T_1) = (q-1)^0 \cdot (-1)^0 \cdot q^2 \cdot \chi_{(1^2)}(T_1) = 1.$$

If $w = s_1$, then $k = 1, w' = s_1 \in S_2, \mu = (1^2), cc_\mu^\lambda = 1, l_\mu^\lambda = 0$, and

$$\chi_{sign}(T_{s_1}) = (q-1)^0 \cdot (-1)^0 \cdot q^2 \cdot \chi_{(1^2)}(T_{s_1}) = -1.$$

If $w = s_1 s_2$, then $k = 3, w' = 1 \in S_0, \mu = (0), cc_\mu^\lambda = 1, l_\mu^\lambda = 2$, and

$$\chi_{sign}(T_{s_1 s_2}) = (q-1)^0 \cdot (-1)^2 \cdot q^2 \cdot \chi_{(0)}(T_1) = 1.$$

On the the last character, the reflection representation character: here $\lambda = (21)$. If $w = 1$, then $k = 1, w' = 1 \in S_2$. There are two choices for μ: (2) and (1^2), and in both cases $cc_\mu^\lambda = 1$, $l_\mu^\lambda = 0$. Thus,

$$\chi_{ref}(T_1) = (q-1)^0 \cdot (-1)^0 \cdot q^0 \cdot \chi_{(2)}(T_1) + (q-1)^0 \cdot (-1)^0 \cdot q^0 \cdot \chi_{(1^2)}(T_1) = 2.$$

If $w = s_1$, then $k = 1, w' = s_1 \in S_2$. There are two choices for μ: (2) and (1^2), and in both cases $cc_\mu^\lambda = 1$, $l_\mu^\lambda = 0$. Thus,

$$\chi_{ref}(T_{s_1}) = (q-1)^0 \cdot (-1)^0 \cdot q^0 \cdot \chi_{(2)}(T_{s_1}) + (q-1)^0 \cdot (-1)^0 \cdot q^0 \cdot \chi_{(1^2)}(T_{s_1}) = q - 1.$$

And finally, if $w = s_1 s_2$, then $k = 3, w' = 1 \in S_0$, μ must equal (0), so $cc_\mu^\lambda = 1, l_\mu^\lambda = 1$. Thus,

$$\chi_{ref}(T_{s_1 s_2}) = (q-1)^0 \cdot (-1)^1 \cdot q^1 \cdot \chi_{(0)}(T_1) = -q.$$

Thus we arrive at the character table for $\mathcal{H}(S_3)$:

$$\begin{array}{c} \mathcal{H}(S_3) \\ \chi_{(3)} \\ \chi_{(1^3)} \\ \chi_{(21)} \end{array} \begin{array}{c} \begin{array}{ccc} T_{w_{(1^3)}} & T_{w_{(21)}} & T_{w_{(3)}} \end{array} \\ \begin{bmatrix} 1 & q & q^2 \\ 1 & -1 & 1 \\ 2 & q-1 & -q \end{bmatrix} \end{array}.$$

B.2 Starkey's Rule

We can calculate the same character table using Starkey's Rule. Recall Starkey's Rule from Section 4.4:

$$\chi(T_{w_\lambda}) = \sum_{\nu \vdash n} \chi(w_\nu) p_\lambda^\nu,$$

where

$$p^\nu_\lambda = \frac{|C_\nu \cap S_\lambda|}{|S_\lambda|} \det(q \cdot id_{V_\lambda} - \rho_\lambda(w_\nu)).$$

We take for granted the character table of S_3:

$$\begin{array}{c c} S_3 & \begin{array}{ccc} w_{(1^3)} & w_{(21)} & w_{(3)} \end{array} \\ \begin{array}{c} \chi_{(3)} \\ \chi_{(1^3)} \\ \chi_{(21)} \end{array} & \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{bmatrix} \end{array}.$$

Here, ρ_λ is the reflection representation of S_λ. The following is the table of matrices $\rho_\lambda(w_\nu)$ (see [GP00, Lemma 1.1.6]):

$$\begin{array}{c c} \lambda\backslash\nu & \begin{array}{ccc} (1^3) & (21) & (3) \end{array} \\ \begin{array}{c} (3) \\ (1^3) \\ (21) \end{array} & \begin{bmatrix} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} & \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} -1 & -1 \\ 1 & \end{bmatrix} \\ * & \emptyset & \emptyset \\ \begin{bmatrix} 1 \end{bmatrix} & \begin{bmatrix} -1 \end{bmatrix} & \emptyset \end{bmatrix} \end{array},$$

where the $*$ refers to the 0×0 matrix.

Thus we may compute the p^ν_λ:

$$\begin{array}{c c} \lambda\backslash\nu & \begin{array}{ccc} (1^3) & (21) & (3) \end{array} \\ \begin{array}{c} (3) \\ (1^3) \\ (21) \end{array} & \begin{bmatrix} \frac{1}{6}(q-1)^2 & \frac{1}{2}(q^2-1) & \frac{1}{3}(q^2+q+1) \\ 1 & 0 & 0 \\ \frac{1}{2}(q-1) & \frac{1}{2}(q+1) & 0 \end{bmatrix} \end{array}.$$

Let P be the matrix of the p^λ_μ values, and let B be the character table of S_3. Then Starkey's Rule tells us that the character table of $\mathcal{H}(S_3)$ is BP^T:

$$\begin{array}{c c} \mathcal{H}(S_3) & \begin{array}{ccc} T_{w_{(1^3)}} & T_{w_{(21)}} & T_{w_{(3)}} \end{array} \\ \begin{array}{c} \chi_{(3)} \\ \chi_{(1^3)} \\ \chi_{(21)} \end{array} & \begin{bmatrix} 1 & q & q^2 \\ 1 & -1 & 1 \\ 2 & q-1 & -q \end{bmatrix} \end{array}.$$

This is (unsurprisingly) the same character table we got using the Murnaghan-Nakayama rule.