Last time: started on repns of sym. gps.
(will continue next time)

Today (interlude): $\underbrace{\text{Repn theory}}_{\text{complex!}}$ of $GL_2(\mathbb{F}_q)$

[Fulton-Harris §5.2]

[Piatetski-Shapiro]

closer connection to repn theory of $GL_n(\mathbb{F}_q)$ & $GL_n(\mathbb{Q}_p)$

Let $k = \mathbb{F}_q$

$G = GL_2(k)$

$B = \left\{ \begin{pmatrix} a & b \\ & d \end{pmatrix} \middle| \ a, d \in k^\times, b \in k \right\} \subseteq G$ Borel subgp.

$T = \left\{ \begin{pmatrix} a & \\ & d \end{pmatrix} \middle| \ a, d \in k^\times \right\} \subseteq B$ maximal (split) torus

$U = \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \middle| \ b \in k \right\} \subseteq B$ unipotent subgp.

We have $B = U \rtimes T$ and $|B| = (q-1)^2 q$

G act transitively on the $q+1$ 1D subspaces of k^2 $\mathbb{P}^1(k)$
and B is the stabilizer of $\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle \subseteq k^2$

So $|G/B| = q+1 \Rightarrow |G| = (q-1)^2 q(q+1)$

Let $L = \mathbb{F}_q(\sqrt{D})$ be the unique quadratic ext'n of K.

$L^\times \cong \left\{ \begin{pmatrix} x & Dy \\ y & x \end{pmatrix} \middle| \begin{array}{l} x, y \text{ not} \\ \text{both } 0 \end{array} \right\} \subseteq G$

If $\varphi = x + y\sqrt{D} \in L$, then

$\mathrm{Tr}\begin{pmatrix} x & Dy \\ y & x \end{pmatrix} = 2x = \mathrm{Tr}_{L/K}\,\varphi, \quad \det\begin{pmatrix} x & Dy \\ y & x \end{pmatrix} = x^2 - Dy^2 = N_{L/K}\,\varphi.$

Conjugacy classes: Jordan form

Four cases:

a) 1 eigenvalue $x \in K$, diagonalizable

b) 1 eigenvalue $x \in K$, not diagonalizable

c) distinct eigenvalues $x, y \in K$ $(x \neq y)$

d) distinct eigenvalues, $x \pm y\sqrt{D} \in L \setminus K$

Representative	Num elts. in class	Num. classes
$a_x = \begin{pmatrix} x & \\ & x \end{pmatrix}$	1	$q-1$
$b_x = \begin{pmatrix} x & 1 \\ & x \end{pmatrix}$	q^2-1	$q-1$
$c_{x,y} = \begin{pmatrix} x & \\ & y \end{pmatrix}$	q^2+q	$\frac{1}{2}(q-1)(q-2)$
$d_{x,y} = \begin{pmatrix} x & Dy \\ y & x \end{pmatrix}$	q^2-q	$\frac{1}{2}(q^2-q)$
Total:	$(q-1)^2 q(q+1)$ elts.	q^2-1 conj. classes

Principal series repns

Parabolic induction

- T is abelian, so all irreps are 1D

$$\mu_{\alpha,\beta}\begin{pmatrix} a & \\ & c \end{pmatrix} := \alpha(a)\beta(c) \qquad \alpha,\beta : k^\times \to \mathbb{C}^\times$$

- "Inflate" to B: $B \to B/U \cong T$

$$\grave{\rho}_{\alpha,\beta}(\underset{\in U}{u}\,\underset{\in T}{t}) := \mu_{\alpha,\beta}(t).$$

- Induce to G:

$$\underbrace{\rho_{\alpha,\beta}}_{\text{might be reducible}} := \mathrm{Ind}_B^G \grave{\rho}_{\alpha,\beta}.$$

Applying Prop 18, its character is

	a_x	b_x	$c_{x,y}$	$d_{x,y}$
$\chi_{\alpha,\beta}$:	$(q+1)\alpha(x)\beta(x)$	$\alpha(x)\beta(x)$	$\alpha(x)\beta(y)+\alpha(y)\beta(x)$	0

So $\rho_{\alpha,\beta} \cong \rho_{\beta,\alpha}$, and otherwise they are nonisom.

One can compute

$$(\chi_{\alpha,\beta}, \chi_{\alpha,\beta}) = \begin{cases} 1, & \text{if } \alpha \neq \beta \\ 2, & \text{if } \alpha = \beta \end{cases}$$

so $\rho_{\alpha,\beta}$ is irred. if $\alpha \neq \beta$.

Let $\rho_{\det,\alpha} : G \to \mathbb{C}$ (q-1) such repn.

$g \mapsto \alpha(\det g)$

This has character (and since it's 1D, equals)

$\chi_{\det,\alpha}: \quad \alpha(x)^2 \qquad \alpha(x)^2 \qquad \alpha(x)\alpha(y) \qquad \alpha(x^2 - Dy^2)$

And we can compute

$(\chi_{\alpha,\alpha}, \chi_{\det,\alpha}) = 1$

So $\rho_{\alpha,\alpha} = \rho_{\det,\alpha} \oplus \tilde{\rho}_\alpha$ ← q-dim.

where $\tilde{\rho}_\alpha$ is the irrep. w/ character

$\tilde{\chi}_\alpha: \quad q\alpha(x)^2 \qquad 0 \qquad \alpha(x)\alpha(y) \qquad -\alpha(x^2 - Dy^2)$

In the "principal series" (induced from 1D repns of B), we have

- $q-1$ irreps $\rho_{\det,\alpha}$ of dim 1
- $q-1$ irreps $\tilde{\rho}_\alpha$ of dim q
- $\frac{1}{2}(q-1)(q-2)$ irreps. $\rho_{\alpha,\beta}$, $\alpha \neq \beta$ of dim $q+1$

Remaining: $\frac{1}{2}(q^2 - q)$ irreps.

Cuspidal repns

We've gotten all the irreps. we can arising from irreps. of

$$T = \left\{ \begin{pmatrix} x & \\ & y \end{pmatrix} \middle| x, y \in K^\times \right\}$$

As a K-v.s.,

$$\left\{ \begin{pmatrix} x & Dy \\ y & x \end{pmatrix} \middle| x, y \in K^\times \right\} \cong T$$

$\rotatebox{90}{$\subseteq$}$ $L^\times$ (non split torus)

So let's look at 1D repns of $L^\times$

There $|L^\times| = q^2-1$ of them

If ρ is a repn of $K^\times$, we obtain a repn of $L^\times$:

$$\underset{\in L}{\tilde{\rho}(\ell)} := \rho\left(N_{L/K}(\ell)\right) \quad (q\text{-}1 \text{ repns of this form})$$

Call a 1D repn of $L^\times$ <u>indecomposable</u> if it doesn't factor through $N_{L/K}$ in this way.

These are exactly the 1D repns ν of $L^\times$ s.t.

$$\overline{\nu}(\ell) := \nu(\overline{\ell}) \quad \text{satisfies } \overline{\nu} \neq \nu$$

We wish we could just induce these repns up, but it's not (nearly) that simple.

Turns out that (HW2?)

$$\tilde{\rho}_1 \otimes \rho_{\alpha,1} \cong \rho_{\alpha,1} \oplus \mathrm{Ind}_{L^\times}^{G} \nu \oplus \rho_\nu$$

where ρ_ν is an irrep. w/ character

$\chi_\nu \quad (q-1)\nu(x) \qquad -\nu(x) \qquad 0 \qquad -\nu(x+y\sqrt{D}) - \bar{\nu}(x+y\sqrt{D})$

Let $W := L^\times \rtimes \underset{\langle\sigma\rangle}{\underset{\parallel}{C_2}}$ (Weil gp.)

$$\sigma l = \bar{l} \sigma$$

Consider the 2D-repns τ of W.

Since $L^\times$ is abelian,

$$\tau|_{L^\times} = \nu_1 \oplus \nu_2$$

Case 1: τ is irred.

Turn out that $\qquad\qquad\qquad \tau_\nu$

$$\bar{\nu}_1 = \nu_2$$

and $\nu := \nu_1$ is incomposable $\qquad \frac{1}{2}q(q-1)$ of these

Case 2: τ is red

Turns out that ν_1 & ν_2 factor through $N_{L/K}$:

$$\nu_i(\ell) = \underbrace{\mu_i}_{\text{some 1D } K^\times \text{ repn.}}(N_{L/K}(\ell))$$

τ_{μ_1,μ_2}

$\frac{1}{2}q(q-1)$ of these

So we have a correspondence btwn. all 2D-repns of W and the higher-dim'l irreps. of G:

$$\tau_\nu \longleftrightarrow \rho_\nu$$

$$\tau_{\alpha,\beta} \longleftrightarrow \begin{cases} \rho_{\alpha,\beta}, & \text{if } \alpha \neq \beta \\ \tilde{\rho}_\alpha, & \text{if } \alpha = \beta \end{cases}$$