Announcements

HW2 first part posted (due Wed. 2/25 @ 9am)

Friday's class cancelled; I will post a recorded lecture instead (Topic: $GL_2(\mathbb{F}_q)$)

Now:

Starting a new unit of symmetric gp. repns.

Sources: Sagan, Fulton-Harris, James (char p)

Today: partitions and tableaux

Let S_n be the symmetric gp. on n letters

The conj. classes of S_n are parametrized by (integer) partitions

$$\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n,$$

which are sequences of integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ s.t. $\lambda_1 + \cdots + \lambda_k = n$.

"parts"

$w \in S_n$ has cycle type λ if λ_i is the size of the ith largest cycle in w $\forall i$.

We want one S_n-irrep for every partition $\lambda \vdash n$.

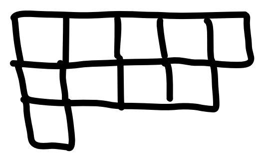
The Young diagram (or Ferrers diagram) for λ is the top-and-left justified array of boxes w/ λ_i boxes in row i.

The length of λ is the number $\ell(\lambda)$ of parts of λ. i.e. the number of rows in the Young diagram for λ.

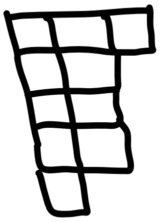
The size (or order) of λ is $|\lambda| = \lambda_1 + \cdots + \lambda_{\ell(\lambda)}$ i.e. the number of boxes in the Young diagram for λ

The conjugate partition of λ is the partition λ' corresp. to the Young diag. of λ reflected over the line $y = -x$.

The number of cols of the Young diag. for λ equals $\lambda_1 = \ell(\lambda')$, and $|\lambda'| = |\lambda|$.

e.g. $\lambda = (5,4,1)$  $\ell(\lambda) = 3$

$|\lambda| = |\lambda'| = 10$

$\lambda' = (3,2,2,2,1)$  $\ell(\lambda') = 5$

Three orders on partitions:

- Containment (Young's lattice): $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i \ \forall i$.

- Dominance order: $\lambda \trianglelefteq \mu$ if $\forall i$,

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i$$

- Lexicographic order: $\lambda \leq \mu$ if $\lambda = \mu$ or $\lambda_i < \mu_i$ for the minimal i s.t. $\lambda_i \neq \mu_i$

(partial order)

$\uparrow$ refines

(partial order)

$\uparrow$ refines

(total order)

Def 20: A <u>(Young) tableau</u> T of <u>shape</u> λ is a filling of the boxes of (the Young diag. of) λ w/ pos. integers.

- T is <u>semistandard</u> if the entries increase weakly along rows and strictly down columns
- T is <u>standard</u> if it is semistandard and has entries $1, 2, \ldots, |\lambda|$, appearing once each

e.g.

$$\begin{array}{|c|c|c|c|c|} \hline 2 & 2 & 3 & 5 & 6 \\ \hline 4 & 5 & 7 & 7 \\ \cline{1-4} 5 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 6 & 8 \\ \hline 3 & 5 & 7 & 10 \\ \cline{1-4} 9 \\ \cline{1-1} \end{array} \qquad \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 4 & 2 & 2 \\ \hline 3 & 5 & 7 & 7 \\ \cline{1-4} 3 \\ \cline{1-1} \end{array}$$

semistandard   standard   neither

Lemma 21 (Dominance Lemma): Let T be a std. tableau of shape λ and S be a std. tableau of shape μ. If the entries in row i of T all live in different cols of S $\forall i$, then $\lambda \trianglelefteq \mu$.

Pf: Sort the cols of S s.t the entries in the first i rows of T all appear in the first i rows of (the sorted) S. Then,

$$\begin{aligned}\lambda_1 + \cdots + \lambda_i &= \#\text{entries in the first } i \text{ rows of } T \\ &\leq \#\text{entries in the first } i \text{ rows of } S \\ &= \mu_1 + \cdots + \mu_i .\end{aligned}$$

$\square$

Def 22: The <u>Young subgroup</u> (or <u>parabolic subgp.</u>) of S_n corresponding to $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash n$ is

$$\begin{aligned}S_\lambda &:= S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k} \\ &= \underbrace{S_{\{1,\ldots,\lambda_1\}}}_{\substack{\text{permutes first} \\ \lambda_1 \text{ integers}}} \times \underbrace{S_{\{\lambda_1+1,\ldots,\lambda_2\}}}_{\substack{\text{permutes next} \\ \lambda_2 \text{ integers}}} \times \cdots \times \underbrace{S_{\{n-\lambda_k+1,\ldots,n\}}}_{\substack{\text{permutes last} \\ \lambda_k \text{ integers}}}\end{aligned}$$

More generally, let T be a std. tableau of shape λ.
The row stabilizer of T is the subgp.

$$R_T := \{ w \in S_n \mid w \text{ preserves the rows of } T \} \subseteq S_n$$

The column stabilizer of T is the subgp.

$$C_T := \{ w \in S_n \mid w \text{ preserves the cols. of } T \} \subseteq S_n$$

We have $R_T \cong S_\lambda$ and $C_T \cong S_{\lambda'}$

Example:

$$T = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & 6 & 8 \\ \hline 3 & 5 & 7 & 10 \\ \cline{1-4} 9 \\ \cline{1-1} \end{array}$$

$$R_T = S_{\{1,2,4,6,8\}} \times S_{\{3,5,7,10\}} \times S_{\{9\}} \cong S_{(2,2,1)}$$

$$C_T = S_{\{1,3,9\}} \times S_{\{2,5\}} \times S_{\{4,7\}} \times S_{\{6,10\}} \times S_{\{8\}} \cong S_{(3,2,2,2,1)}$$

Def 23: For any finite gp. G, the <u>group algebra</u> $\mathbb{C}[G]$ is the v.s. w/ basis indexed by the elts. of g and multiplication induced from g.

e.g. $\mathbb{C}[S_3] = \langle (), (12), (13), (23), (123), (132) \rangle$

$$[() + (13)][(123) - 3(12)] = (123) - 3(12) + (23) - 3(123)$$

(Note that the G-action on $\mathbb{C}[G]$ is the regular repn.)

Any G-repn. can be extended to a $\mathbb{C}[G]$-module by linearity:

$$\rho(ag+bh) := a\rho(g) + b\rho(h)$$

e.g. if

$$\rho((12)) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \quad \rho((123)) = \begin{bmatrix} & -1 \\ 1 & -1 \end{bmatrix},$$

then

$$\rho\left((12)+(123)\right) = \begin{bmatrix} & 0 \\ 2 & -1 \end{bmatrix}$$