Today: Restriction / induction, Frobenius reciprocity
[F-H §3.3] [Serre §3.3]

Recall: Def 17b): Let $H \leq G$.

Let $\sigma_1, \ldots, \sigma_k$ be a set of representatives for G/H.

If (π, W) is an H-repn, the <u>induced repn</u> of (ρ, W) to G is the G-repn

$$\mathrm{Ind}_H^G(\pi, W) = (\rho, V)$$

where

$$V := \bigoplus_{i=1}^{k} W_i$$

and

$$\rho(g)\overset{\in W_i}{w_i} = (\pi(h_i)w)_j$$

where $\underset{\in G/H}{g\sigma_i = \sigma_j} \underset{\in H}{h}$

Ind_H^G doesn't depend on the choice of coset reps. (up to isom.)

Pf: Homework #2, or Frobenius reciprocity (later today) $\square$

Also note that

- $$\dim \mathrm{Ind}_H^G W = |G:H| \dim W$$

and

$$\mathrm{Res}_H^G \mathrm{Ind}_H^G W \cong |G:H| W$$

- $\mathrm{Ind}_H^G (W_1 \oplus W_2) = \mathrm{Ind}_H^G W_1 \oplus \mathrm{Ind}_H^G W_2$

- $\mathrm{Ind}_K^G \mathrm{Ind}_H^K W = \mathrm{Ind}_H^G W$

Ex: a) $V_{reg} = \mathrm{Ind}_1^G V_{triv}$

b) More generally,

$$\mathrm{Ind}_H^G V_{triv}$$

is the <u>permutation repn.</u> (HW1 #3) of the action of

G on G/H: $\underset{\substack{\in \\ G/H}}{\sigma} \underset{\substack{\in \\ h}}{h} \cdot v_\tau := v_{\sigma\tau}$

e.g. $H = (12) \subseteq S_3$

Coset reps: $\sigma_1 = ()$, $\sigma_2 = (13)$ $\sigma_3 = (23)$

$V := \mathrm{Ind}_H^G V_{triv} = \langle v_{()}, v_{(13)}, v_{(23)} \rangle$

$() \mapsto Id_V$

$(12)v_{()} = v_{()}$

Corrected

$(12)v_{(13)} = (12)(13)v_{()} = (123)v_{()} = (23)(12)v_{()} = (23)v_{()} = v_{(23)}$

$(12)v_{(23)} = (12)(23)v_{()} = (132)v_{()} = (13)(12)v_{()} = (13)v_{()} = v_{(13)}$

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$= (23)(12)v_{()} = (23)v_{()} = v_{(23)}$

$(123)v_{(13)} = (123)(13)v_{()} = (23)v_{()} = v_{(23)}$

etc.

Now, fix G/H coset representatives $\sigma_1, \ldots, \sigma_k$, taking $\sigma_1 = 1$. As an H-repn., $hw_1 = (hw)_1$, so we can identify W with $W_1 \subseteq \mathrm{Ind}_H^G W$.

Proposition 18: Let $V = \mathrm{Ind}_H^G W$. Then,

$$\chi_V(g) = \frac{1}{|H|} \sum_{\substack{a \in G \\ a^{-1}ga \in H}} \chi_W(a^{-1}ga)$$

Pf: Since $V = \bigoplus_{i=1}^{k} W_i$, $\chi_V(g) = \sum_i \mathrm{Tr}\, g|_{W_i}$.

Let $g\sigma_i = \sigma_j h$, $h \in H$, and if $i \neq j$, then $\mathrm{Tr}_{W_i} g = 0$. The cases where $g\sigma_i = \sigma_i h$ are exactly those i where $\sigma_i^{-1} g \sigma_i \in H$, and in that case $gW_i \subseteq W_i$ and $gw_i = (\sigma_i^{-1} g \sigma_i w)_i$, so

$\mathrm{Tr}\, g|_{W_i} = \mathrm{Tr}\, \sigma_i^{-1} g \sigma_i|_W$. Thus, $\chi_V(g) = \sum_{\sigma_i^{-1} g \sigma_i \in H} \chi_W(\sigma_i^{-1} g \sigma_i)$.

The formula follows from the fact that if $\sigma_i^{-1} g \sigma_i \in H$ and $\sigma_i H = aH$, then $a^{-1}ga \in H$ and $\chi_W(a^{-1}ga) = \chi_W(\sigma_i^{-1} g \sigma_i)$.

$\square$

Theorem 19 (Frobenius reciprocity): Let W be an H-repn and V be a G-repn.

There is a natural v.s. equivalence

$$\mathrm{Hom}_H(W, \mathrm{Res}_H^G V) \cong \mathrm{Hom}_G(\mathrm{Ind}_H^G W, V).$$

Equivalently,

$$(\chi_W, \chi_{\mathrm{Res}_H^G V})_H \cong (\chi_{\mathrm{Ind}_H^G W}, \chi_V)_G.$$

If W, V: irred, the mult. of W in $\mathrm{Res}_H^G V$ equals the mult. of V in $\mathrm{Ind}_H^G W$.

Pf: The equivalence of the two statements, and the last statement as a consequence or the second, are by Cor. 12 (or just by Schur's Lemma).

For the first statement:

- If $\phi \in \mathrm{Hom}_G(\mathrm{Ind}_H^G W, V)$, then

$$\phi|_{W_1} \in \mathrm{Hom}_H(W, \mathrm{Res}_H^G V).$$

• If $\varphi \in \mathrm{Hom}_H(W, \mathrm{Res}^G_H V)$, then $\phi \in \mathrm{Hom}_G(\mathrm{Ind}^G_H W, V)$, where

$$\phi|_{W_i} := \sigma_i \varphi \sigma_i^{-1},$$

indep. of choice of reps. since $\sigma_i h \varphi h^{-1} \sigma_i = \sigma_i \varphi \sigma_i$

$h\varphi = \varphi h$

$$w_i \mapsto w \mapsto \varphi(w) \mapsto \sigma_i[\varphi(w)]$$

corrected

This is G-equivariant since if $g \in G$ satisfies

$g\sigma_i = \sigma_j h$, $h \in H$, then

$$g\phi(w_i) = g\,\sigma_i[\varphi(w)] = \sigma_j h[\varphi(w)] = \sigma_j[\varphi(hw)]$$

H-equivariance.

$$= \phi([hw]_j) = \phi(g \cdot w_i).$$

$\square$