Last time: character orthogonality

Today: colum orthogonality, examples

Prop 15: The number of irreps. of G equals the number of conj. classes of G.

Pf: By Cor. 11, we already know $\leq$.

For $\geq$, suppose $\alpha \in \mathbb{C}_{cl}(G)$ and (α, χ_V) for all irreps V. We will show that $\alpha = 0$, and the resalt follows.

For any G-repn V, set

$$\varphi_V := \sum_{g \in G} \alpha(g)\, \rho_V(g) \in \operatorname{End} V$$

One can check that in fact $\varphi_V \in \operatorname{End}_G V$.

[F-H Prop. 2.28]

By Schur's Lemma, $\varphi_V = \lambda \cdot \mathrm{Id}$, so

$$\lambda = \frac{1}{\dim V} \operatorname{Tr}(\varphi_V) = \frac{1}{\dim V} \sum_{g \in G} \alpha(g)\, \chi_V(g)$$

$$= \frac{|G|}{\dim V} (\alpha, \chi_{V^*}) = 0.$$

So $0 = \varphi_V = \sum_{g\in G} \alpha(g)\rho_V(g)$ for any G-repn V.

But if $V = V_{reg}$, then $\{\rho_V(g), g\in G\}$ are linearly independent matrices (pf: $\rho_V(g)v_e = v_g$, and the v_g's are linearly indep. vectors). Therefore, $\alpha = 0$. $\square$

Cor 16: The irred. chars. form an orthonormal basis of $\mathbb{C}$. Furthermore, the columns of the character table are also orthogonal:

$$\sum_{\chi \text{ irred}} \overline{\chi(g)}\chi(h) = \begin{cases} |\text{centralizer of } g|, & \text{if } g = h \\ 0, & \text{otherwise} \end{cases}$$

Pf: The first statement follows from Thm. 10 and Prop 15. Column orthogonality is a consequence of squareness and row orthogonality. The details are left as an exercise for the reader (and potentially HW2). $\square$

Example: S_4

	1	6	8	6	3
	$()$	(12)	(123)	(1234)	$(12)(34)$
χ_{triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
χ_{ref}	3	1	0	-1	-1
$\chi_{ref} \otimes \chi_{sgn}$	3	-1	0	1	-1
χ_W	2	0	-1	0	2

Notes:

$V_{perm} = \mathbb{C}^4$ $\quad \omega \cdot v_i := v_{\omega(i)}$ has character:

χ_{perm}: 4 2 1 0 0

decomposes as χ_{triv} + (something), and use inner product to show that ^ is irred.

for W: use column orthogonality

What about A_4?

	1	4	4	3
	$()$	(123)	(132)	$(12)(34)$
χ_{triv}	1	1	1	1
χ_{ref}	3	0	0	-1
χ	1	ρ	ρ^2	1
χ'	1	ρ^2	ρ	1

Notes:

χ_W	2	-1	-1	2

has $(\chi_W, \chi_W) = 2$

use orthog. to compute the characters.

We obtained characters of A_4 by "restricting" characters of S_4. Wouldn't it be nice to go the other way?

Def 17: Let $H \leq G$.

a) If (ρ, V) is a G-repn, the <u>restriction</u> of (ρ, V) to H is the H-repn.

$$\mathrm{Res}_H^G(\rho, V) = (\rho|_H, V). \qquad \begin{pmatrix} \text{or } \mathrm{Res}_H^G V \\ \text{or } \mathrm{Res}_H^G \rho \end{pmatrix}$$

b) Let $\sigma_1, \ldots, \sigma_k$ be a set of representatives for G/H.

If (π, W) is an H-repn, the <u>induced repn</u> of (ρ, W) to G is the G-repn

$$\mathrm{Ind}_H^G(\pi, W) = (\rho, V)$$

where

$$V := \bigoplus_{i=1}^{k} W_i$$

and

$$\rho(g)\underbrace{w_i}_{\in W_i} = (\pi(h_i)w)_j$$

where $g\underset{\in G/H}{\sigma_i} = \sigma_j \underset{\in H}{h}$

Ind_H^G doesn't depend on the choice of coset reps. (up to isom.)

Also note that

$$\dim \mathrm{Ind}_w^G W = |G:H| \dim W$$

and

$$\mathrm{Res}_H^G \mathrm{Ind}_H^G W \cong |G:H| W$$

Ex:

a) $V_{reg} = \mathrm{Ind}_1^G V_{triv}$

b) More generally,

$$\mathrm{Ind}_H^G V_{triv}$$

is the <u>permatation repn.</u> (HW1 #3) of the action of G on G/H: $\underset{\in G/H}{\sigma}\underset{\in h}{h} \cdot v_\tau := v_{\sigma\tau}$

Next time: characters of induced repns, and Frobenius reciprocity