Last time: character theory basics

Today: character orthogonality, decomposition of the regular repn [FH §2.2, Serre §2.3–2.4]

For any G-repn V, let V^G be the isotypic component of the trivial repn.

Prop 9: The map

$$\varphi := \frac{1}{|G|} \sum_{g \in G} g \in \operatorname{End}_G V$$

is a projection $V \to V^G$. Furthermore, the multiplicity of the trivial repn inside V is given by

$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g). \qquad (*)$$

Pf: If $v \in V$, then for all $h \in G$,

$$h\varphi(v) = \frac{1}{|G|} \sum_{g \in G} hgv = \frac{1}{|G|} \sum_{g' \in G} g'v = \varphi(v),$$

so $\operatorname{im} \varphi \subseteq V^G$. On the other hand, if $v \in V^G$, then

$$\varphi(v) = \frac{1}{|G|} \sum_{g \in G} v = v, \text{ so } V^G \subseteq \operatorname{im} \varphi \text{ and } \varphi^2 = \varphi.$$

For the mult. of the trivial repn, since φ is a projection,

$$\dim V^G = \mathrm{Tr}(\varphi) = \frac{1}{|G|}\sum_{g\in G}\mathrm{Tr}(g) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g).$$

$\square$

We'll use this to prove orthogonality of characters

Let $\mathbb{C}_{cl}(G)$ be the space of class functions $G \to \mathbb{C}$.

We have a Hermitian inner product on $\mathbb{C}_{cl}(G)$:

$$(\alpha, \beta) := \frac{1}{|G|}\sum_{g\in G}\overline{\alpha(g)}\,\beta(g)$$ (Convention note: linear in <u>second</u> spot)

Thm 10: The characters of G-irreps. form an orthonormal set in $\mathbb{C}_{cl}(G)$.

Pf: Let V, W be G-irreps. We apply the previous result to $\mathrm{Hom}(V,W)$. By HW1 #1a and Prop. 8,

$$\chi_{\mathrm{Hom}(V,W)} = \chi_{V^*\otimes W} = \overline{\chi_V}\,\chi_W$$

On the other hand, by HW 1 #1b, $\mathrm{Hom}(V,W)^G = \mathrm{Hom}_G(V,W)$.
By Schur's Lemma, we have

$$\dim \mathrm{Hom}_G(V,W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

Applying (*) now gives

$$(\chi_V, \chi_W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{if } V \not\cong W. \end{cases}$$

$\square$

Corollary 11: The number of irreps. of G is $\leq$ the number of conjugacy classes $\square$

Corollary 12: If V, W have irred. decomps.

$$V \cong c_1 V_1 \oplus \cdots \oplus c_k V_k,$$

$$W \cong d_1 V_1 \oplus \cdots \oplus d_k V_k,$$

then $(\chi_V, \chi_W) = \dim \mathrm{Hom}_G(V,W) = \sum_i c_i d_i$.

In particular,

- if V is irred, its multiplicity in W is (χ_V, χ_W),
- V is irred. $\iff (\chi_V, \chi_V) = 1$.
- Every repn. is determined by its character.

Let V_{reg} be the regular repn of G (see HW1 #4)

Cor 13: Every G-irrep V appers in V_{reg} with multiplicity $\dim V$. That is, if $V_1, \ldots, V_k$ are the G-irreps,

$$V_{reg} \cong \bigoplus_i (\dim V_i) V_i \qquad (**)$$

Pf: Since the action of a nonidentity elt. has no fixed pts.,

$$\chi_{reg}(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{otherwise.} \end{cases}$$

If V is an irrep, the multiplicity of V in G is

$$(\chi_V, \chi_{reg}) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)}\, \chi_{reg}(g) = \dim V.$$

$\square$

Taking the character of both sides of $(**)$, we get

Cor 14:

$$|G| = \sum_i (\dim V_i)^2,$$

and for $g \neq e$

$$\sum_i (\dim V_i)\, \chi_{V_i}(g) = 0.$$

Special case of <u>column orthogonality</u>

$\square$

Prop 15: The number of irreps. of G equals the number of conj. classes of G.

Pf: By Cor. 11, we already know $\leq$.
For $\geq$, suppose $\alpha \in \mathbb{C}_{cl}(G)$ and (α, χ_V) for all irreps V. We will show that $\alpha = 0$, and the resalt follows.
For any G-repn V, set

$$\varphi_V := \sum_{g \in G} \alpha(g)\, \rho_V(g) \in \operatorname{End} V$$

One can check that in fact $\varphi_V \in \operatorname{End}_G V$.

[F-H Prop. 2.28]

By Schur's Lemma, $\varphi_V = \lambda \cdot \mathrm{Id}$, so

$$\lambda = \frac{1}{\dim V} \operatorname{Tr}(\varphi_V) = \frac{1}{\dim V} \sum_{g \in G} \alpha(g)\, \chi_V(g)$$

$$= \frac{|G|}{\dim V} (\alpha, \chi_{V^*}) = 0.$$

So $0 = \varphi_V = \sum_{g \in G} \alpha(g) \rho_V(g)$ for any G-repn V.

But if $V = V_{reg}$, then $\{\rho_V(g), g \in G\}$ are linearly independent matrices (Pf: $\rho_V(g)v_e = v_g$, and the v_g's are linearly indep. vectors). Therefore, $\alpha = 0$. $\square$

Cor 16: The irred. chars. form an orthonormal basis of $\mathcal{C}$. Furthermore, the columns of the character table are also orthogonal:

$$\sum_{\chi \text{ irred}} \overline{\chi(g)}\,\chi(h) = \begin{cases} |\text{centralizer of } g|, & \text{if } g = h \\ 0, & \text{otherwise} \end{cases}$$

Pf: The first statement follows from Thm. 10 and Prop 15. Column orthogonality is left as an exercise to the reader (and potentially HW2). $\square$