<u>Crystal bases</u> [Hong-Kang, Bump-Schilling]

Two desiderata:

1) In a highest wt. repn. V^λ, the raising/lowering operators send wt. spaces into wt. spaces. We would like to extend this to the basis vectors themselves by choosing a basis for each wt. space V^λ_μ s.t. the raising/lowering operators send basis vectors to (scalar multiples of) basis vectors or 0.

For $\mathfrak{gl}_n/\mathfrak{sl}_n$, natural indexing set since

$$\dim V^\lambda_\mu = K_{\lambda\mu} = \#\text{ semistd. tableaux of shape } \lambda \text{ \& content } \mu$$

For $\mathfrak{sl}_2$, this choice is possible since all wt. spaces are 1-dim.

For general $\mathfrak{g}$, not possible for most repns. e.g.

$$\begin{tikzcd} {} \arrow[dr, "f_i"'] & & {} \arrow[dl, "f_j"] \\ & \mathbb{C}^2 & \\ {} \arrow[ur, "e_i"] & & {} \arrow[ul, "e_j"'] \end{tikzcd}$$

usually won't lead to a compatible choice of basis for $\mathbb{C}^2$

Same issue for $U_q(\mathfrak{g})$ with generic q.

2) In a tensor product $V^{\otimes k}$ of copies of the std. repn., we would like the decomposition into irreps. to partition the basis vectors

e.g. $U_q(\mathfrak{sl}_2)$, $V = V_q^{(1)} = \mathrm{span}\{v_+, v_-\}$

Irred. decomp:

$$V \otimes V \cong V_q^{(2)} \oplus V_q^{(0)}$$

where $V_q^{(2)} = \mathrm{span}(v_{++},\ v_{+-} + q v_{-+},\ v_{--})$

$V_q^{(0)} = \mathrm{span}(v_{-+} - q v_{+-})$

v_{+-} and v_{-+} lie in neither subrepn...

...unless we can set $q = 0$

Thm 104 (Lusztig, Kashiwara): For any reductive Lie algebra $\mathfrak{g}$, and for every highest wt. repn. V_q^λ of $U_q(\mathfrak{g})$, after taking an appropriate limit $q \to 0$, there exists a basis of the resulting repn. satisfying 1).

In many cases (and always in type A), it also satisfies 2).

This is called the <u>crystal basis</u> for V^λ or V_q^λ

The <u>Kashiwara operators</u> $\tilde{e}_i, \tilde{f}_i$ send basis vectors to basis vectors

The <u>crystal</u> graph has vertices the basis elts. and edges $v \xrightarrow{i} w$ when $\tilde{f}_i(v) = w$.

Let's work in the case $\mathfrak{g} = \mathfrak{sl}_n$.

$$V := V^{(1)} = \text{span}\{\text{semistd. tableaux w/ shape } \square \text{ and entry } \le n\}$$
$$= \text{span}\{\boxed{1}, \boxed{2}, \dots, \boxed{n}\}$$

The crystal graph is

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \xrightarrow{3} \cdots \xrightarrow{n-1} \boxed{n}$$

$$\tilde{f}_i(\boxed{j}) = \begin{cases} \boxed{j+1}, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \qquad \tilde{e}_i(\boxed{j}) = \begin{cases} \boxed{j-1}, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

In a tensor product $V^{\otimes k}$,

$$\tilde{f}_i(\boxed{a_1} \otimes \cdots \otimes \boxed{a_k})$$

is given by:

- write) under each i, (under each $i+1$
- Change the rightmost unpaired i to an $i+1$

e.g.

$$\tilde{f}_2(2 \otimes 1 \otimes 3 \otimes 3 \otimes 2 \otimes 1 \otimes 2 \otimes 2 \otimes 3 \otimes 1 \otimes 2)$$
$$\begin{array}{ccccccccccc}) & & (& (&) & &) &) & (& &) \end{array}$$

$$= 2 \otimes 1 \otimes 3 \otimes 3 \otimes 2 \otimes 1 \otimes 3 \otimes 2 \otimes 3 \otimes 1 \otimes 2)$$
$$\begin{array}{ccccccccccc}) & & (& (&) & & (&) & (& &) \end{array}$$

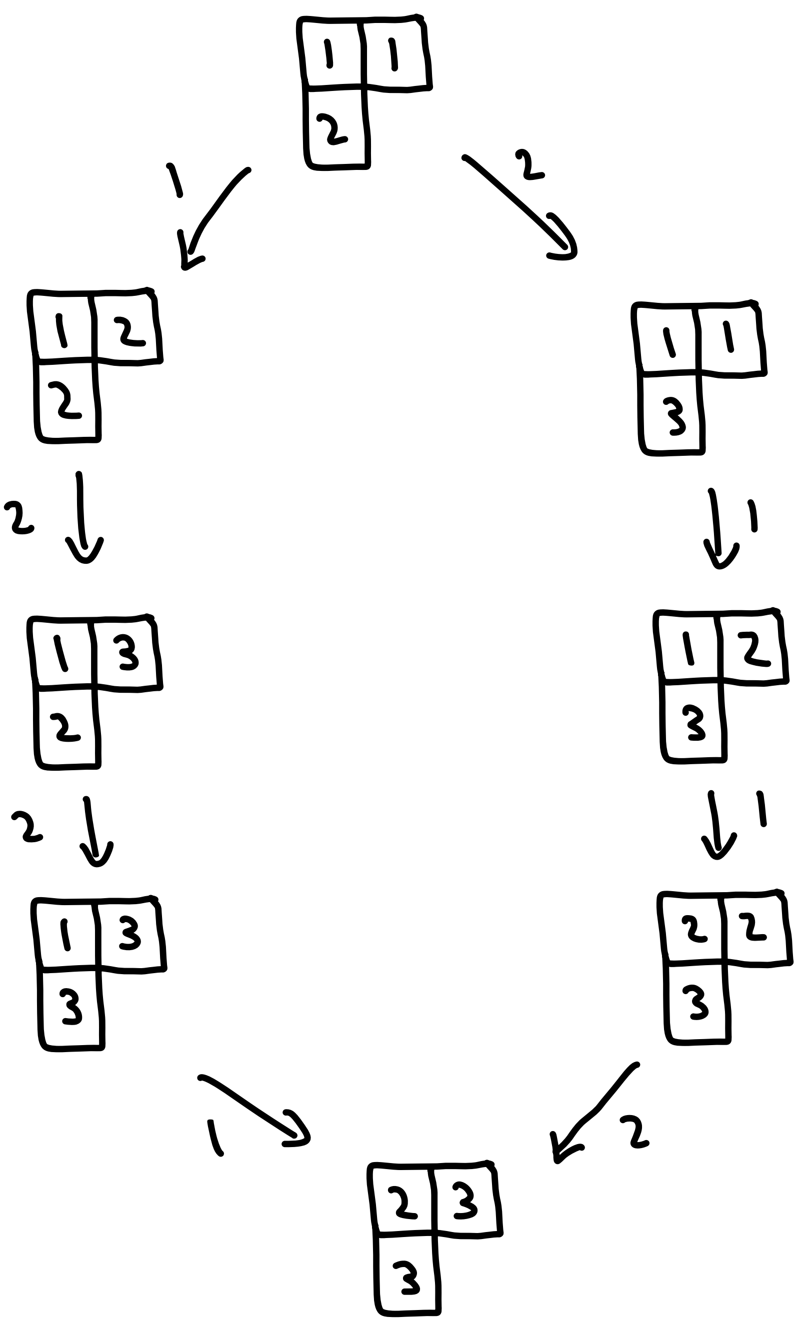
Tableaux of shape λ fit into this framework by <u>row reading</u>

e.g

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \longmapsto \boxed{3} \otimes \boxed{1} \otimes \boxed{2}$$

and we obtain a crystal graph for any V^λ

e.g. $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = (2,1)$.



Tensor product rule allows us to decompose the tensor product of any two highest wt. repns.

E.g. $\mathfrak{g} = \mathfrak{sl}_2 \quad V = V^{(2)}, \ W = V^{(3)}$

$$V : \boxed{1|1} \longrightarrow \boxed{1|2} \longrightarrow \boxed{2|2}$$

$$W : \boxed{1|1|1} \longrightarrow \boxed{1|1|2} \longrightarrow \boxed{1|2|2} \longrightarrow \boxed{2|2|2}$$

$V \otimes W$:

$$\begin{tikzcd}
 & & 11 \otimes 112 \arrow[d] & 11 \otimes 111 \arrow[d] & \\
11222 \arrow[d] & 11122 \arrow[l] \arrow[d, "\times" marking] & 11112 \arrow[l] \arrow[d, "\times" marking] & 11111 \arrow[l] \arrow[d, "\times" marking] & 12 \otimes 111 \arrow[dl] \\
12222 \arrow[d] & 12122 \arrow[l, "\times" marking] \arrow[d] & 12112 \arrow[l] \arrow[d, "\times" marking] & 12111 \arrow[l] \arrow[d, "\times" marking] & \\
22222 & 22122 \arrow[l, "\times" marking] & 22112 \arrow[l, "\times" marking] & 22111 \arrow[l] &
\end{tikzcd}$$

$$V \otimes W \cong V^{(5)} \oplus V^{(3)} \oplus V^{(1)}$$

Thank you all for your hard work,
and a wonderful semester.
Have a great summer!