Announcements

HW1 posted

due Wed. 2/11 @ 9:00am via Gradescope

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course code: email me

Last time: Schur's Lemma & Maschke's Thm

Today: repns of abelian gps.

intro to character theory [F-H §2.1, Serre §2.1]

Proposition 6: If G is abelian, then every G-irrep. is one-dimensional.

Pf: First we claim that $\rho(g) \in \mathrm{End}_G(V)$ $\forall g \in G$.

Indeed, if $h \in G$, $v \in V$, then since G is abelian,

$$\rho(g)\rho(h)v = \rho(h)\rho(g)v,$$

so $\rho(g)$ is G-equivariant.

By Schur's Lemma, every $\rho(g)$ is a scalar mult. of the identity. This means that every subspace of V is G-invariant, so V must be 1-dim. $\square$

Ex: $G = \mathbb{Z}/n\mathbb{Z} = \langle g \rangle$.

Irreps of G are of the form

$$\rho_\zeta : G \to \mathbb{C}^* \qquad \zeta : n\text{th root of } 1$$
$$g^a = \zeta^a$$

So there are exactly n nonisom. irreps. of G!

$G=\mathbb{Z}/4\mathbb{Z}$	$\rho(1)$	$\rho(g)$	$\rho(g^2)$	$\rho(g^3)$
ρ_1	1	1	1	1
ρ_i	1	i	-1	$-i$
ρ_{-1}	1	-1	1	-1
ρ_{-i}	1	$-i$	-1	i

Very nice looking table! (Square, roots of unity, orthog. rows/cols.)

And tells us everything we'd want to know about the repn theory of G

Goal: get as close as we can to this for all finite gps.

Def 7: The <u>character</u> of a repn. (ρ, V) is the function $\chi_V : G \to \mathbb{C}$ given by the trace of ρ:

$$\chi_V(g) = \mathrm{Tr}(\rho(g)).$$

Prop 8:

a) The character is a <u>class function</u>:

$$\chi_V(hgh^{-1}) = \chi_V(g)$$

b) $\chi_V(1) = \dim V$

c) $\chi_{V \oplus W} = \chi_V + \chi_W$

d) $\chi_{V \otimes W} = \chi_V \chi_W$

e) $\chi_{V^*} = \overline{\chi_V}$

f) $\chi_{\mathrm{Sym}^2 V}(g) = \frac{1}{2}\left[\chi_V(g)^2 + \chi_V(g^2)\right]$

g) $\chi_{\Lambda^2 V}(g) = \frac{1}{2}\left[\chi_V(g)^2 - \chi_V(g^2)\right]$

Pf: a) Trace is a class function

b) $\rho(1) = Id_V$

c), d), e) Let $g \in G$, and let $\lambda_1, \ldots, \lambda_k$ be the e-values of $\rho_V(g)$, and $\mu_1, \ldots, \mu_\ell$ be the e-values of $\rho_\ell(g)$. Then,

c) The e-values of $\rho_{V \oplus W}(g) = \rho_V(g) \oplus \rho_W(g)$ are $\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_\ell$

d) The e-values of $\rho_{V \otimes W}(g) = \rho_V(g) \otimes \rho_W(g)$ are $\lambda_i \mu_j$, $1 \leq i \leq k$, $1 \leq j \leq \ell$

e) The e-values of $\rho_{V^*}(g) = \rho_V(g^{-1})^T$ are $\lambda_1^{-1}, \ldots, \lambda_k^{-1}$. Since $\rho_V(g)$ has finite order, its e-values are roots of unity, so $\lambda_i^{-1} = \overline{\lambda_i}$, and so

$$\chi_{V^*}(g) = \overline{\lambda_1} + \ldots + \overline{\lambda_k} = \overline{\chi_V(g)}$$

f), g) See HW1

$\square$

We collect the characters of irreps. of G into the <u>character table</u>

- Rows indexed by irreps.
- Cols indexed by conjugacy classes in G
- Entries are the character value for the irrep. at (any elt. of) the conj. class.

Ex: Recall from Lecture 1 the three irreps of S_3: (we'll see soon that there are no more)

$$\rho_{triv} : w \longmapsto [1]$$

$$\rho_{sgn} : w \longmapsto [(-1)^w]$$

$$\rho_{ref} : () \longmapsto \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \qquad (23) \longmapsto \begin{bmatrix} 1 & -1 \\ & -1 \end{bmatrix}$$

$$(12) \longmapsto \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \qquad (123) \longmapsto \begin{bmatrix} & -1 \\ 1 & -1 \end{bmatrix}$$

$$(13) \longmapsto \begin{bmatrix} -1 & \\ -1 & 1 \end{bmatrix} \qquad (132) \longmapsto \begin{bmatrix} -1 & 1 \\ -1 & \end{bmatrix}$$

Conj. classes are det'd by cycle type

So the character table is:

S_3	1 elt $()$	3 elts (12)	2 elts (123)
χ_{triv}	1	1	1
χ_{sgn}	1	-1	1
χ_{ref}	2	0	-1

Class activity! Compute the characters of the following repns? Can they be written as sums of irred. chars.?

a) $\rho_{triv} \oplus \rho_{sgn}$

b) ρ_{ref}^*

c) $\rho_{ref}^{\otimes 2}$

d) $\mathrm{Sym}^2 \rho_{ref}$

e) $\Lambda^2 \rho_{ref}$

f) The <u>regular repn</u> $V_{reg} = \langle v_g \mid g \in S_3 \rangle$

w/ the action $\underset{\in S_3}{w} \cdot v_u := v_{wu}$