Last time: $U_q(\mathfrak{sl}_2)$ is a non-cocomm. Hopf alg.

i.e. $\tau : V \otimes W \to W \otimes V$ is not an isom.

$a \otimes b \mapsto b \otimes a$

Today: Replacement for that condition.

Let $V = V_q^{(1)} = \mathbb{C}^2$ be the $U_q(\mathfrak{sl}_2)$-irrep. w/ highest-wt 1 (std. repn.)

We have $V = \mathrm{span}\{v_+, v_-\}$ where

$$K v_\pm = q^{\pm 1} v_\pm \qquad \begin{matrix} E v_+ = 0 & F v_+ = v_- \\ E v_- = v_+ & F v_- = 0 \end{matrix}$$

$V \otimes V$ has basis $\{v_{++}, v_{+-}, v_{-+}, v_{--}\}$

where $v_{\pm\pm} = v_\pm \otimes v_\pm$

Let $\hat{R} : V \otimes V \to V \otimes V$ be the map

$$\hat{R} = \begin{array}{cc} \begin{matrix} v_{++} & v_{+-} & v_{-+} & v_{--} \end{matrix} & \\ \begin{bmatrix} q & & & \\ & & 1 & \\ & 1 & q-q^{-1} & \\ & & & q \end{bmatrix} & \begin{matrix} v_{++} \\ v_{+-} \\ v_{-+} \\ v_{--} \end{matrix} \end{array}$$

If $q \mapsto 1$, $\hat{R} \mapsto \tau$.

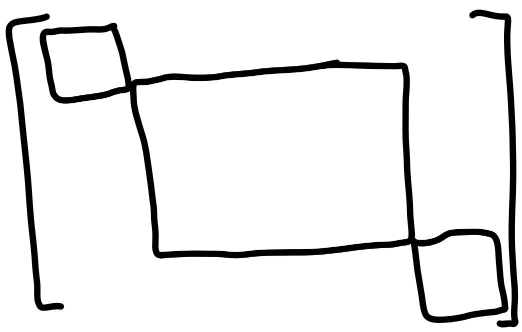
Prop 100: $\hat{R}$ is a $U_q(\mathfrak{sl}_2)$-module isom. That is,

$$\hat{R} \circ \Delta(x) = \Delta(x) \circ \hat{R} \qquad \text{for all } x \in U_q(\mathfrak{sl}_2)$$

Pf: $\hat{R}$ is bijective because it's nonsingular. Since $\Delta(K) = K \otimes K$ is symmetric $\hat{R}$ commutes w/ $\Delta(K)$ if and only if it preserves the wt. spaces. These are:

span$\{v_{++}\}$	weight q^2
span$\{v_{+-}, v_{-+}\}$	weight 1
span$\{v_{--}\}$	weight q^{-2}

This holds if $\hat{R}$ has the block diagonal form



,

which it does.

Since $\hat{R}$ acts by a scalar on 1-dim wt. spaces, we only need to check that it commutes w/ $\Delta(E)$ and $\Delta(F)$ on span$\{v_{+-}, v_{-+}\}$.

We have

$$\Delta(E)\hat{R}(v_{+-}) = \Delta(E)(v_{-+}) = (1\otimes E + E\otimes k)(v_{-+}) = 0 + qV_{++}$$

$$\hat{R}\Delta(E)(v_{+-}) = \hat{R}(1\otimes E + E\otimes k)(v_{+-}) = \hat{R}(V_{++}+0) = qV_{++}$$

$$\Delta(E)\hat{R}(v_{-+}) = (1\otimes E + E\otimes k)(v_{+-} + (q-q^{-1})v_{-+})$$

$$= V_{++} + 0 + 0 + q(q-q^{-1})v_{++} = q^2 V_{++}$$

$$\hat{R}\Delta(E)(v_{-+}) = \hat{R}(0 + qV_{++}) = q^2 v_{++}$$

And similarly for $\Delta(F)$ $\square$

In fact, $\hat{R}$ satisfies the <u>Yang-Baxter equation</u>:

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \in \mathrm{End}(V^{\otimes 3})$$

where $\hat{R}_{ij}$ acts as R on the ith and jth tensor factors.

More commonly, this is stated:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in \mathrm{End}(V^{\otimes 3})$$

where $R = \tau\hat{R}$.

In addition, by direct computation, we obtain the <u>quadratic rel'n</u>: $(\hat{R}-q)(\hat{R}+q^{-1}) = 0$

Def 101: A Hopf algebra H is <u>quasicocommutative</u> if there exists an invertible element $R \in H \otimes H$ s.t

$$\tau\Delta(x) = R\Delta(x)R^{-1} \quad \forall x \in H.$$

H is quasitriangular if R further satisfies

$$(\Delta \otimes id)R = R_{13}R_{23}, \qquad (id \otimes \Delta) = R_{13}R_{12}$$

This R is called the <u>universal R-matrix</u>, and satisfies the YBE $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$

Thm 102 (Drinfeld): Let $H = U_q(\mathfrak{sl}_2)$. The matrix

$$R = q^{(H \otimes H)/2} \sum_{n \geq 0} \frac{(q-q^{-1})^n}{[n]_q!} q^{n(n-1)/2} E^n \otimes F^n \in H \hat{\otimes} H,$$

where $q^{(H\otimes H)/2}$ acts on the wt. space $V_k \otimes V_\ell$ by $q^{k\ell/2}$

and $[n]_q! = [n]_q [n-1]_q \cdots [1]_q$,

satisfies the conditions of Def 101. $\square$

Similarly $U_q(\mathfrak{g})$ is "essentially quasitriangular" for any reductive Lie algebra $\mathfrak{g}$

Similar to $U_q(\mathfrak{sl}_2)$, $H = U_q(\mathfrak{gl}_n)$ has a highest-wt. repn.

V_q^λ which is a deformation of the $U(\mathfrak{gl}_n)$-repn V^λ.

Let $V = V_q^{(1)} = \mathbb{C}^n$, the "standard" $U_q(\mathfrak{gl}_n)$ repn.

$U_q(\mathfrak{gl}_n)$ acts on $V^{\otimes k}$ via $\Delta^{(k)} = \Delta \circ \cdots \circ \Delta : H \to H^{\otimes k}$

The braid group $B_k = \langle \sigma_1, \ldots, \sigma_{k-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
$\sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| \geq 2 \rangle$

also acts on $V^{\otimes k}$ via $\sigma_i \mapsto \hat{R}_{i,i+1}$.

In fact $\hat{R}$ also satisfies the quad. reln. $(\hat{R} - q)(\hat{R} + q^{-1}) = 0$,
so we obtain a repn. of the <u>Hecke algebra</u>

$$\mathcal{H}_k(q) = \langle T_1, \ldots, T_{k-1} \mid T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},\ T_i T_j = T_j T_i,\ |i-j| \geq 2,\ (T_i - q)(T_i + q^{-1}) \rangle$$

via $T_i \mapsto \hat{R}_{i,i+1}$.

Thm 103 (Jimbo): The quantum group $U_q(\mathfrak{gl}_n)$ and the Hecke algebra $\mathcal{H}_k(q)$ are mutual centralizers inside $V^{\otimes k}$. We have the $U_q(\mathfrak{gl}_n) \otimes \mathcal{H}_k(q)$-module decomposition

$$V^{\otimes k} \cong \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} V_q^\lambda \otimes S_q^\lambda, \text{ where}$$

S_q^λ is the $\mathcal{H}_k(q)$-irrep. deforming the Specht module S^λ.