Quantum Groups [Kassel], [Chari-Pressley], [Hong-Kang]

Recall: $\mathfrak{sl}_2$ has generators e, f, h w/ rel'ns

$$[h,e] = 2e, \qquad [h,f] = -2f, \qquad [e,f] = h.$$

The <u>universal enveloping algebra</u> $U(\mathfrak{sl}_2)$ is the assoc. $\mathbb{C}$-alg. w/ generators e, f, h and rel'ns

$$he - eh = 2e, \quad hf - fh = -2f, \quad ef - fe = h$$

Poincaré-Birkhoff-Witt Thm: $\{e^a f^b h^c \mid a,b,c \geq 0\}$ forms a basis

(similar construction for other Lie algebras)

Def 97:

a) A <u>bialgebra</u> (over $\mathbb{C}$) is an assoc. unital algebra H w/ mult. $\mu : H \otimes H \to H$ and unit $\iota : \mathbb{C} \to H$, along with algebra homoms. $\Delta : H \to H \otimes H$ (comultiplication) and $\varepsilon : H \to \mathbb{C}$ (counit) satisfying

- Coassociativity: $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$
- Counit: $(\varepsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon) \circ \Delta$
- μ and ι are <u>coalgebra homoms.</u>: maps $\phi : C' \to C$ satisfying $(\phi \otimes \phi) \circ \Delta_{C'} = \Delta_C \circ \phi$

$$\varepsilon_C \circ \phi = \varepsilon_{C'}$$

(w/out μ and ι, this defines a <u>coalgebra</u>)

b) A <u>Hopf algebra</u> is a bialgebra along with a linear map $S: H \to H$ (antipode) satisfying (S is an alg. antiautom.)

$$\mu \circ (S \otimes id) \circ \Delta = \iota \circ \varepsilon = \mu \circ (id \otimes S) \circ \Delta$$

c) Let τ be the flip map $a \otimes b \mapsto b \otimes a$.

A bialg. or Hopf alg. is <u>cocommutative</u> if $\tau \circ \Delta = \Delta$

Examples:

a) $H = \mathbb{C}[G]$ (group algebra)

$\Delta(g) = g \otimes g$, $\quad \varepsilon(g) = 1 \quad$ $S(g) = g^{-1}$

Check:

Coassoc: $(\Delta \otimes id) \circ \Delta(g) = g \otimes g \otimes g = (id \otimes \Delta) \circ \Delta(g)$

Antipode rel'n:

$$\begin{aligned} \mu \circ (S \otimes id) \circ \Delta(g) &= \mu(S \otimes id)(g \otimes g) \\ &= \mu(g^{-1} \otimes g) = 1_{e_G} = \iota \circ \varepsilon \end{aligned}$$

H is cocomm: $\tau \circ \Delta(g) = g \otimes g = \Delta(g)$

b) $H = U(\mathfrak{g})$ $\mathfrak{g}$: Lie alg.

$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x, \quad x \in \mathfrak{g}$

Check:

Coassoc: $(\Delta \otimes id) \circ \Delta(x) = (\Delta \otimes id)(x \otimes 1 + 1 \otimes x)$

$= x \otimes 1 \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x$

$= (id \otimes \Delta) \circ \Delta(x)$

Counit: $(\varepsilon \otimes id) \circ \Delta(x) = (\varepsilon \otimes id)(x \otimes 1 + 1 \otimes x) = x$

H is cocomm: $T \circ \Delta(\ ) = 1 \otimes x + x \otimes 1 = \Delta(x)$

If H is a Hopf alg. and U, V, W are H-modules, then

- H has a "trivial repn": $h \cdot \underset{\in \mathbb{C}}{z} := \varepsilon(h) z$
- V^* is an H-module via $(h \cdot \varphi)(v) := \varphi(S(x) \cdot v)$
- $V \otimes W$ is an H-module via $h \cdot v \otimes w := \Delta(h)(v \otimes w)$
- By coassoc., $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$
- If H is cocomm, then $\tau : V \otimes W \to W \otimes V$ is an isom.

What about a Hopf algebra that <u>isn't</u> cocomm.?

Fix $q \in \mathbb{C}^\times$ or transcendental.

Def 98: The quantum gp. $U_q(\mathfrak{sl}_2)$ is the assoc. $\mathbb{C}$-alg. generated by E, F, K, K^{-1} w/ rel'ns

$$KK^{-1} = K^{-1}K = 1$$

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F$$

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

(Set $K = q^h$ and send $q \to 1$ to recover $U(\mathfrak{sl}_2)$)

$U_q(\mathfrak{sl}_2)$ is a Hopf algebra:

$$\Delta(K) = K \otimes K \qquad \varepsilon(K) = 1 \qquad S(K) = K^{-1}$$

$$\Delta(E) = 1 \otimes E + E \otimes K \qquad \varepsilon(E) = 0 \qquad S(E) = -EK^{-1}$$

$$\Delta(F) = K^{-1} \otimes F + F \otimes 1 \qquad \varepsilon(F) = 0 \qquad S(F) = -KF$$

Not cocommutative!

Generically, the repn. theory of $U_q(\mathfrak{sl}_2)$ is very similar to that of $U(\mathfrak{sl}_2)$ or $\mathfrak{sl}_2$.

Thm 99: Suppose that $q \in \mathbb{C}^\times$ is not a root of unity. The irred. f.d. repns of $U_q(\mathfrak{sl}_2)$ are parametrized by nonneg. ints. k. The irrep $V^{(k)}$ has a basis of K-evectors $v_k, v_{k-2}, \ldots, v_{-k}$ such that

$$K v_j = q^j v_j$$

$$E v_{k-2l} = [l]_q v_{k-2l+2}$$

$$F v_{k-2l} = [k-l]_q v_{k-2l-2}$$

where

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}} = \underbrace{q^{n-1} + q^{n-3} + \cdots q^{-(n-1)}}_{n \text{ terms}}$$

Ex: a) Std. repn ($k=1$): $V^{(1)} = \mathbb{C}^2$

$$K \mapsto \begin{bmatrix} q & \\ & q^{-1} \end{bmatrix} \quad E \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad F \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

b) $k=2$:

$$K \mapsto \begin{bmatrix} q^2 & & \\ & 1 & \\ & & q^{-2} \end{bmatrix} \quad E \mapsto \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & q+q^{-1} \\ 0 & 0 & 0 \end{bmatrix} \quad F \mapsto \begin{bmatrix} 0 & 0 & 0 \\ q+q^{-1} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

If time: check rel'ns