Quiver Representations

[Schiffler] [Etingof et. al. Ch. 6]

Def 93:

a) A <u>quiver</u> $Q = (Q_0, Q_1, s, t)$ is a directed graph w/

Q_0: vertices $\quad s: Q_1 \to Q_0$ "source"

Q_1: arrows $\quad t: Q_1 \to Q_0$ "target" $\quad s(\alpha) \xrightarrow{\alpha} t(\alpha)$

b) A <u>repn</u> of Q is an assignment of:

- to each vertex w, a v.s. V_w
- to each arrow $v \xrightarrow{\alpha} w$, a linear map $V_\alpha : V_v \to V_w$

(we will restrict to finite quivers and f.d. $\mathbb{C}$-v.s.)

c) A <u>morphism</u> $V \to W$ of Q repns is a collection of maps

$$\varphi_v : V_v \to W_v, \quad v \in Q_0,$$

that commute w/ the linear maps:

$$\begin{tikzcd} V_v \arrow[r, "V_\alpha"] \arrow[d, "\varphi_v"'] \arrow[dr, phantom, "\circlearrowright"] & V_w \arrow[d, "\varphi_w"] \\ W_v \arrow[r, "W_\alpha"'] & W_w \end{tikzcd} \qquad \forall\ v \xrightarrow{\alpha} w$$

Isomorphism, subrepn, quotient repn., direct sum, are defined similarly as for gps., Lie algebras, etc.

d) A Q-repn V is <u>simple</u> (or <u>irreducible</u>) if its only subrepns of V and $\{0\}$. V is <u>indecomposible</u> if

$V_v = \{0\}, V_\alpha = 0$

$V \cong V' \oplus V'' \Rightarrow V'$ or $V'' = 0$.

By the Krull-Schmidt Theorem, every f.d. quiver repn can be written as a direct sum of indecomposibles, unique up to isom./ordering.

Examples:

a) Vertex: $Q = \bullet$

repns: $\mathbb{C}^n$

only one indecomposible: $\mathbb{C}$ (also simple)

b) Arrow: $Q = \bullet \to \bullet$

repns: $\mathbb{C}^m \xrightarrow{A} \mathbb{C}^n$ $\quad A \in \mathrm{Mat}_{n,m}$

Choosing bases of $\mathbb{C}^m$, $\mathbb{C}^n$, can write

$$A = \underbrace{I_r}_{r := \operatorname{rank} A} \oplus 0 \qquad \begin{array}{ccc} \mathbb{C}^r & \xrightarrow{I} & \mathbb{C}^r \\ \oplus & & \oplus \\ \mathbb{C}^{m-r} & & \mathbb{C}^{n-r} \\ \ker A & & \operatorname{coker} A \end{array}$$

Indecomposibles: $\mathbb{C} \to 0$, $\quad 0 \to \mathbb{C}$, $\quad \mathbb{C} \xrightarrow{1} \mathbb{C}$

$S(1) \qquad S(2) \qquad P(1)$

not simple!

Morphisms:

$$\begin{tikzcd}
S(2) \arrow[d, hook] & 0 \arrow[r] \arrow[d] & \mathbb{C} \arrow[d, "1"] \\
P(1) \arrow[d] & \mathbb{C} \arrow[r, "1"] \arrow[d, "1"'] & \mathbb{C} \arrow[d] \\
S(1) & \mathbb{C} \arrow[r] & 0
\end{tikzcd}$$

c) Loop: $Q = \begin{tikzcd} \bullet \arrow[loop below] \end{tikzcd}$ repns: $\begin{tikzcd} \mathbb{C}^m \arrow[loop below, "A"] \end{tikzcd}$ $A \in \operatorname{End}(\mathbb{C}^m)$

indecomposibles $\longleftrightarrow$ Jordan blocks

$$J_{\lambda,n} = \underbrace{\begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}}_{n} \qquad \begin{array}{l} \lambda \in \mathbb{C} \\ n \geq 1 \end{array}$$

d) Kronecker quiver $\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$ Repns: $\begin{tikzcd} V \arrow[r, bend left, "A"] \arrow[r, bend right, "B"'] & W \end{tikzcd}$

Indecomps. w/ $\dim V = \dim W = 1$ are

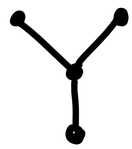
$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] & \mathbb{C} \end{tikzcd}$ a, b not both 0 up to mutual scalar i.e. pts $[a:b]$ in $\mathbb{P}^1(\mathbb{C})$

e) A_n Dynkin diag., any orientation:

e.g. $\bullet \rightarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet \leftarrow \bullet \leftarrow \bullet$

Indecomposibles correspond to intervals:

$$V(i,j): \quad \overset{1}{0} - 0 - - 0 - \overset{i}{\mathbb{C}} \xrightarrow{1} \mathbb{C} - - \xrightarrow{1} \overset{j}{\mathbb{C}} - 0 - - - \overset{n}{0}$$

f) D_4 Dynkin diagram 

Simples:

$$\begin{tikzcd} \mathbb{C} \arrow[r] & 0 & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] & 0 & \mathbb{C} \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] & 0 & 0 \arrow[l] \\ & \mathbb{C} \arrow[u] & \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] & \mathbb{C} & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd}$$

Other indecomposibles:

$$\begin{tikzcd} \mathbb{C} \arrow[r, "1"] & \mathbb{C} & 0 \arrow[l] \\ & 0 \arrow[u] & \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] & \mathbb{C} & \mathbb{C} \arrow[l, "1"'] \\ & 0 \arrow[u] & \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] & \mathbb{C} & 0 \arrow[l] \\ & \mathbb{C} \arrow[u, "1"] & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "1"] & \mathbb{C} & \mathbb{C} \arrow[l, "1"'] \\ & 0 \arrow[u] & \end{tikzcd} \quad \begin{tikzcd} \mathbb{C} \arrow[r, "1"] & \mathbb{C} & 0 \arrow[l] \\ & \mathbb{C} \arrow[u, "1"] & \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] & \mathbb{C} & \mathbb{C} \arrow[l, "1"'] \\ & \mathbb{C} \arrow[u, "1"] & \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{C} \arrow[r, "1"] & \mathbb{C} & \mathbb{C} \arrow[l, "1"'] \\ & \mathbb{C} \arrow[u, "1"] & \end{tikzcd} \quad \begin{tikzcd} \mathbb{C} \arrow[r, "{\begin{bmatrix}1\\0\end{bmatrix}}"] & \mathbb{C}^2 & \mathbb{C} \arrow[l, "{\begin{bmatrix}0\\1\end{bmatrix}}"'] \\ & \mathbb{C} \arrow[u, "{\begin{bmatrix}1\\1\end{bmatrix}}"'] & \end{tikzcd}$$

Three subspaces:

$$\begin{tikzcd} U \arrow[r, hook] & U+V+W & V \arrow[l, hook'] \\ & W \arrow[u, hook] & \end{tikzcd}$$

$$\dim(U+V+W) = \dim U + \dim V + \dim W - \dim(U\cap V) - \dim(U\cap W) - \dim(V\cap W) + \dim(U\cap V\cap W) + \#\left(\begin{tikzcd} \mathbb{C} \arrow[r] & \mathbb{C}^2 & \mathbb{C} \arrow[l] \\ & \mathbb{C} \arrow[u] & \end{tikzcd}\right)$$

(mathoverflow/23478: "Most common mathematical false belief")