Invariant theory questions:

Q1: Is $\mathbb{C}[V]^G$ finitely generated?

Q2: What is a set of generators?

First Fundamental Theorem (FFT)

Q3: What are the relations btwn. these generators?

Second Fundamental Theorem (SFT)

For GL_n, Q1: yes

Q2: generated by contractions

Q3: today

Also today: finite gp invariant theory (Molien series)

Recall: $W = V^p \oplus (V^*)^q \qquad GL_n = GL(V) \curvearrowright W$

$$g \cdot \underbrace{(v_1, \dots, v_p, \varphi_1, \dots, \varphi_q)}_{w = (v, \varphi)} = (gv_1, \dots, gv_p, \varphi_1 \circ g^{-1}, \dots, \varphi_q \circ g^{-1})$$

Identify $V^p \cong \mathrm{Hom}(\mathbb{C}^p, V)$, $(V^*)^q \cong \mathrm{Hom}(V, \mathbb{C}^q)$

$e_i \mapsto v_i \qquad v \mapsto (\varphi_1(v), \dots, \varphi_q(v))$

Define

$$\mu : W \mapsto \mathrm{Mat}_{q,p}(\mathbb{C})$$

$$\mu(v, \varphi) = \varphi \circ v : \mathbb{C}^p \xrightarrow{v} V \xrightarrow{\varphi} \mathbb{C}^q \in \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q)$$

$$\mu(v,\varphi) = \begin{bmatrix} \varphi_1(v_1) & \cdots & \varphi_1(v_p) \\ \varphi_2(v_1) & \ddots & \varphi_2(v_p) \\ \vdots & & \vdots \\ \varphi_q(v_1) & \cdots & \varphi_q(v_p) \end{bmatrix}$$

The pullback is

$$\mu^*: \mathbb{C}[\mathrm{Mat}_{q,p}] \to \mathbb{C}[W]$$

$$f \longmapsto f \circ \mu$$

$$\mu^*(\underbrace{z_{ij}}_{\substack{\text{coord.}\\ \text{fun}}}) = z_{ij} \circ \mu = (j|i)$$

By the FFT, $\mu^*: \mathbb{C}[\mathrm{Mat}_{q,p}] \to \mathbb{C}[W]^{GL_n}$ is surjective.

Thm 88 (SFT for GL_n): The kernel of μ^* is the ideal $I_{n+1} \subseteq \mathbb{C}[\mathrm{Mat}_{q,p}]$ gen'd by $(n+1)\times(n+1)$ minors. In particular, if $n \geq \min(q,p)$, then the contractions are alg. indep.

We'll need:

Thm 89: I_{n+1} is a radical ideal corresp. to the determinantal variety $D^{(n)}_{q,p} = \{z \in \mathrm{Mat}_{q,p} \mid \operatorname{rank} z \leq n\}$

We claim that $\operatorname{im}\mu = D^{(n)}_{q,p}$. Since $\mu(v,\varphi)$ factors through V, its rank must be $\leq n$. Conversely, if $Z \in \mathrm{Mat}_{q,p}$ has rank $\leq n$, then it can be seen to factor into $Z = XY$ w/ $X \in \mathrm{Mat}_{q,n}$, $Y \in \mathrm{Mat}_{n,p}$, so $Z \in \operatorname{im}\mu$.

(If Z has rank $r \leq n$, can write

$$Z = U \underbrace{\begin{bmatrix} I_r & \\ & 0 \end{bmatrix}}_{q \times p} V, \qquad U \in GL_q,\ V = GL_p.$$

Take $X = U \underbrace{\begin{bmatrix} I_r & \\ & 0 \end{bmatrix}}_{q \times n} \in \mathrm{Mat}_{q,n}$, $Y = \underbrace{\begin{bmatrix} I_r & \\ & 0 \end{bmatrix}}_{n \times p} V \in \mathrm{Mat}_{n,p}$)

Now, $f \in \ker(\mu^*)$ iff $f \circ \mu = 0$ i.e. f vanishes on $\operatorname{im}\mu = D^{(n)}_{q,p}$. Thus, by the Nullstellensatz,

$$\ker(\mu^*) = I(D^{(n)}_{q,p}) = \sqrt{I_{n+1}},$$

and by Thm. 89, $\sqrt{I_{n+1}} = I_{n+1}$. $\square$

Cor 90: The invariant ring of GL_n acting on $V^p \oplus (V^*)^q$ is isom to the coord ring of the determinantal variety:

$$\mathbb{C}[V^p \oplus (V^*)^q]^{GL_n} \cong \mathbb{C}[\mathrm{Mat}_{q,n}]/I_{n+1} = \mathbb{C}[D^{(n)}_{q,p}]$$

Invariant theory of finite groups

For a graded $\mathbb{C}$-algebra $R = \bigoplus_{d>0} R_d$ w/ $\dim R_d < \infty$, the Hilbert series is

$$H(R,t) = \sum_{d\geq 0} (\dim R_d) t^d.$$

Thm 91 (Molien's formula): Let G be a finite gp, and let (ρ, V) be a repn. Then,

$$H(\mathbb{C}[V]^G, t) = \frac{1}{|G|} \sum_{g\in G} \frac{1}{\det(I - t\rho(g))}$$

Pf: Since $\mathbb{C}[V] = \mathrm{Sym}(V^*)$, the degree-$d$ piece is $\mathbb{C}[V]_d = \mathrm{Sym}^d(V^*)$, and each $g\in G$ acts on V^* by $\varphi \mapsto \varphi \circ \rho(g)^{-1}$ (contragredient rep'n)

By Proposition 9 (or character orthogonality), the multiplicity of the trivial repn. in $\mathrm{Sym}^d(V^*)$ is

$$\dim \mathrm{Sym}^d(V^*)^G = \frac{1}{|G|} \sum_{g\in G} \chi_{\mathrm{Sym}^d(V^*)}(g)$$

If g acts on V^* w/ evalues $\alpha_1, \dots, \alpha_n$, then

$\chi_{\mathrm{Sym}^d(V^*)}(g) = \sum_{|B|=d} \alpha_1^{B_1} \cdots \alpha_n^{B_n} = h_d(\alpha_1, \dots, \alpha_n)$. Summing over all d, we get

$$\sum_{d \geq 0} h_d(\alpha_1, \dots, \alpha_n) t^d = \prod_{i=1}^{n} \frac{1}{1 - \underbrace{\alpha_i t}_{\text{evalues of}}} = \frac{1}{\det(I - t \underbrace{g|_{V^*}}_{\rho^*(g) = \rho(g^{-1})^T})}, \quad \text{and}$$

doesn't affect det

Summing over all $g \in G$,

$$H(\mathbb{C}[V]^G, t) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I - t\rho(g^{-1}))}$$

replacing g w/ g^{-1}

$$= \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(I - t\rho(g))}$$

$\square$

Example:

$G = \mathbb{Z}/n\mathbb{Z} \curvearrowright \mathbb{C}$ by $a \mapsto e^{2\pi i a/n} =: \rho^a$.

By Molien's formula,

$$H(\mathbb{C}[x]^G, t) = \frac{1}{n} \sum_{a=0}^{n-1} \frac{1}{1 - \rho^a t} = \frac{1}{1 - t^n}.$$

Direct computation shows that $\mathbb{C}[x]^G = \mathbb{C}[x^n]$ ✓

Thm 92 (Chevalley–Shephard–Todd): Let G be a finite subgp of $GL(V)$, $V \cong \mathbb{C}^n$. Then $\mathbb{C}[V]^G$ is a poly. ring iff G is generated by <u>pseudoreflections</u>, non-id. elts. fixing a codim-1 subspace of V.

If this holds,

$$\mathbb{C}[V]^G \cong \mathbb{C}[\underbrace{f_1, \dots f_k}_{\text{alg. indep.}}], \quad \text{we have } k = n,$$

$|G| = d_1 \cdots d_n$, and $H(\mathbb{C}[V]^G, t) = \prod_{i=1}^{n} \frac{1}{1-t^{d_i}}$

$d_i := \deg(f_i)$

Examples:

a) $G = S_n = \langle s_i = (i, i+1) \rangle$, $V = \mathbb{C}^n$ (std. repn.)

$\mathbb{C}[V]^G = \mathbb{C}[e_1, \dots, e_n]$ $\deg e_i = i$,

$|G| = 1 \cdots n$ and $H(\mathbb{C}[V]^G, t) = \prod_{i=1}^{n} \frac{1}{1-t^i}$

b) $G = \mathbb{Z}/2\mathbb{Z} = \{I, -I\}$, $V = \mathbb{C}^2$

$-I$ fixes only $\{0\}$ $\rightsquigarrow$ not a pseudoref'n.

So CST $\Rightarrow$ $\mathbb{C}[V]^G$ is not a poly. ring.

Invariants: Polys in $\mathbb{C}[x, y]$ w/ all terms of even total deg.

So $\mathbb{C}[V]^G \cong \mathbb{C}[\underset{\substack{\parallel \\ a}}{x^2}, \underset{\substack{\parallel \\ b}}{xy}, \underset{\substack{\parallel \\ c}}{y^2}] \cong \mathbb{C}[a,b,c]/(ac-b^2)$

not a poly ring!

Molien's formula:

$$H(\mathbb{C}[V]^G, t) = \frac{1}{2}\left[\frac{1}{(1-t)^2} + \frac{1}{(1+t)^2}\right] = \frac{1+t^2}{(1-t^2)^2}$$