Invariant theory questions:

Q1: Is $\mathbb{C}[V]^G$ finitely generated?

Q2: What is a set of generators?

First Fundamental Theorem (FFT)

Q3: What are the relations btwn. these generators?

Second Fundamental Theorem (SFT)

Last time:

QI : yes for reductive gps

Today : Q2 for GL_n and other classical gps.

Let $V = \mathbb{C}^n$. V and V^* are GL_n-repns:

$$g \cdot v := gv \quad v \in V \qquad g \cdot \varphi := \varphi \circ g^{-1}, \ \varphi \in V^*$$

So GL_n acts on $W = V^p \oplus (V^*)^q = V \oplus \cdots \oplus V \oplus V^* \oplus \cdots \oplus V^*$ by

$$g \cdot \underbrace{(v_1, \ldots, v_p, \varphi_1, \ldots, \varphi_q)}_{w} = (gv_1, \ldots, gv_p, \varphi_1 \circ g^{-1}, \ldots, \varphi_q \circ g^{-1})$$

For $1 \leq i \leq p$, $1 \leq j \leq q$, the <u>contraction</u> is the map

$$(i|j) : W \longrightarrow \mathbb{C}$$
$$w \longmapsto \varphi_j(v_i)$$

$(i|j) \in \mathbb{C}[W]^{GL_n}$ since

$$g \cdot (i|j)(w) = (i|j)(g^{-1}w) = (\varphi_j \circ g)(g^{-1}w) = \varphi_j(v_i) = (i|j)(w)$$

Thm 85 (FFT for GL_n): $\mathbb{C}[W]^{GL_n}$ is generated as a $\mathbb{C}$-alg. by the contractions $(i|j)$.

We'll start with the special case of <u>multilinear</u> functions:

$$f(\lambda_1 v_1, \ldots, \lambda_p v_p, \mu_1 \varphi_1, \ldots, \mu_q \varphi_q) = \lambda_1 \cdots \lambda_p \mu_1 \cdots \mu_q f(w)$$

which are elts. in $\left(V^{\otimes p} \otimes (V^*)^{\otimes q}\right)^*$

Lemma 86: The multilinear elements of $\mathbb{C}[W]^{GL_n}$ satisfy:

a) If $p \neq q$, the only such function is 0.

b) If $p = q$, they are spanned by $\{f_\sigma \mid \sigma \in S_p\}$, where

$$f_\sigma(w) = \prod_{i=1}^{p} \varphi_{\sigma(i)}(v_i)$$

i.e. $f_\sigma = (\sigma(1)|1) \cdots (\sigma(p)|p)$

Pf: Let $g = \lambda I \in GL_n$, $\lambda \in \mathbb{C}^\times$. Then if $f \neq 0$ is multilinear,

$$(g \cdot f)(w) = f(\lambda^{-1} w) = \lambda^{q-p} f(w).$$

GL_n-invariance thus requires $p = q$.

If $p=q$, then the multilinear funs. on W are

$(U \otimes U^*)^* \cong \mathrm{End}(U)$, where $U = V^{\otimes p}$

and $GL_n \subseteq GL(U)$ acts on $\mathrm{End}(U)$ by conjugation.

The isom. is $\alpha : \mathrm{End}(U) \to (U \otimes U^*)^*$,

$\alpha(A)(u \otimes \phi) = \phi(Au)$, which is $GL(U)$-equiv.,

hence GL_n-equiv.

Now, if $f \in \mathbb{C}[W]^{GL_n}$, by Prop 72,

$$f \in \mathrm{End}_{GL_n}(V^{\otimes p}) = \mathrm{span}\{\sigma \mid \sigma \in S_p\} \subseteq \mathrm{End}(V^{\otimes p})$$

Finally, if $\sigma \in \mathrm{End}(V^{\otimes p})$, then

$$\begin{aligned}
&\alpha(\sigma)(v_1 \otimes \cdots \otimes v_p \otimes \varphi_1 \otimes \cdots \otimes \varphi_p) \\
&= (\varphi_1 \otimes \cdots \otimes \varphi_p)(\sigma(v_1 \otimes \cdots \otimes v_p)) \\
&= (\varphi_1 \otimes \cdots \otimes \varphi_p)(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(p)}) \\
&= (\varphi_1 \mid v_{\sigma^{-1}(1)}) \cdots (\varphi_p \mid v_{\sigma^{-1}(p)}) \\
&= f_\sigma(v_1 \otimes \cdots \otimes v_p \otimes \varphi_1 \otimes \cdots \otimes \varphi_p)
\end{aligned}$$

$\square$

Pf sketch of Thm 85:

A function f is <u>multihomogeneous</u> if $\exists d_1, \dots, d_p, e_1, \dots, e_q$ s.t.

$$f(\lambda_1 v_1, \dots, \lambda_p v_p, \mu_1 \varphi_1, \dots, \mu_q \varphi_q) = \lambda_1^{d_1} \cdots \lambda_p^{d_p} \mu_1^{e_1} \cdots \mu_q^{e_q} f(\omega)$$

Any $f \in \mathbb{C}[W]$ is the sum of its multihomog. components

Any multihomog. $f \in \mathbb{C}[W]^{GL_n}$ of degrees $d_1, \dots, d_p, e_1, \dots, e_q$, w/ $D = \sum d_i$, $E = \sum e_i$, has a unique <u>polarization</u>, a G-inv. multilinear fun.

$$\tilde{f} \in \mathbb{C}\left[V^D \oplus (V^*)^E\right]^{GL_n}$$

that satisfies the <u>restitution</u>:

$$\tilde{f}(\underbrace{v_1, \dots, v_1}_{d_1}, \dots, \underbrace{v_p, \dots, v_p}_{d_p}, \underbrace{\varphi_1, \dots, \varphi_1}_{e_1}, \dots, \underbrace{\varphi_q, \dots, \varphi_q}_{d_q}) = f(v_1, \dots, v_p, \varphi_1, \dots, \varphi_q)$$

by Lemma 86, $\tilde{f}$ is generated by polys. in the contractions $(i|j)$, so by restitution, so is f.

$\square$

Other FFT's

Thm 87:

a) Let $V = \mathbb{C}^n$ and let $(v, w) := v \cdot w$. Then $\mathbb{C}[V^k]^{O(n)}$ is generated by the quadratic polys. (v_i, v_j), $1 \leq i \leq j \leq k$.

b) Let $V = \mathbb{C}^{2n}$ and let $\langle v, w \rangle := v \cdot \overline{w}$. Then $\mathbb{C}[V^k]^{Sp_{2n}(\mathbb{C})}$ is generated by the quadratic polys. $\langle v_i, v_j \rangle$, $1 \leq i \leq j \leq k$.

c) Let GL_n act on $\mathrm{End}(\mathbb{C}^n)^m$ by simultaneous conjugation:

$$g \cdot (A_1, \dots, A_m) = (g A_1 g^{-1}, \dots, g A_m g^{-1}).$$

Then $\mathbb{C}[\mathrm{End}(V)^m]^{GL_n}$ is generated by elts

$$\mathrm{Tr}(A_{i_1} \cdots A_{i_r}) \quad 1 \leq i_j \leq n, \quad r \leq n^2.$$

(Special case of Thm 85)