Announcement:
HW5 posted (due Wed. 5/6)

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Invariant theory [Goodman-Wallach Ch. 5] [Kraft-Procesi]

Let $V$ be a f.d. $\mathbb{C}$-v.s. and $G \to GL(V)$ be a repn.
$G$: finite gp. or $\mathbb{C}$-matrix Lie gp. e.g., $GL_n$, $SL_n$, $O(n)$, $Sp_{2n}$

This induced a $G$-repn on the coord ring $\mathbb{C}[V] := \mathrm{Sym}(V^*)$ via

$$(g \cdot f)(v) = f(g^{-1}v), \quad g \in G,\ f \in \mathbb{C}[V],\ v \in V.$$

If $f$ is homog. of deg $d$, so is $g \cdot f$.

Def 82: $f \in \mathbb{C}[V]$ is an <u>invariant</u> if $g \cdot f = f \ \forall g \in G$

The <u>ring of invariants</u> is the subalg. of invariants

$$\mathbb{C}[V]^G := \{ f \in \mathbb{C}[V] \mid g \cdot f = f \ \forall g \in G \}$$

Remark: More generally, if $X$ is an affine variety and $G \curvearrowright X$, then we similarly have a $G$-action $G \curvearrowright \mathbb{C}[X]$, which is sometimes a repn.

Examples:

a) $G = S_n$, $V = \mathbb{C}^n = \operatorname{span}\{v_1, \dots, v_n\}$

$S_n \curvearrowright V$ by permuting coords: $w \cdot v_i = v_{w^{-1}(i)}$

$\mathbb{C}[V] = \mathbb{C}[x_1, \dots, x_n]$ where $x_i(v_j) := \delta_{ij}$

The ring of invariants is

$$\mathbb{C}[x_1, \dots, x_n]^{S_n} = \begin{matrix}\text{ring of symm.} \\ \text{polys. in } x_1, \dots, x_n\end{matrix} \cong \mathbb{C}[e_1, \dots, e_n]$$

elem. sym. polys.

Fun. Thm. of Sym. Funs.
or see Lecture 31

c) $G = SL_n$, $V = \operatorname{Mat}_n(\mathbb{C})$

$SL_n \curvearrowright V$ by left. mult. $g \cdot X := gX$

We claim

$$\mathbb{C}[\operatorname{Mat}_n]^{SL_n} = \mathbb{C}[\det]$$

$$\det = \sum_w (-1)^w z_{1,w(1)} \cdots z_{n,w(n)} \in \mathbb{C}[V]$$

On one hand, $\det(gX) = \det(g)\det(X) = \det(X)$,
so $\det \in \mathbb{C}[\operatorname{Mat}_n]^{SL_n}$

Conversely, if $f \in \mathbb{C}[\operatorname{Mat}]^{SL_n}$, then define $p \in \mathbb{C}[t]$,
$p(t) := f\left(\begin{bmatrix} t & & \\ & 1 & \\ & & \ddots \end{bmatrix}\right)$. Then if $A \in GL_n$, $\det A = t$,

$$f(A) = f\Bigg(\underset{\underset{\in SL_n}{\nearrow}}{g}\begin{pmatrix} t & & \\ & 1 & \\ & & \ddots \\ & & & 1\end{pmatrix}\Bigg) = f\begin{pmatrix} t & & \\ & 1 & \\ & & \ddots \\ & & & 1\end{pmatrix} = p(t)$$

Since $GL_n$ is (Zariski) dense in $\mathrm{Mat}_n$, $f(A) = p(\det A)$ $\forall A$ by continuity.

c) $G = O(n) = \{g \in GL_n \mid gg^T = I\}$, $V = \mathbb{C}^n$ left. mult.

$$\mathbb{C}[V]^{O(n)} = \mathbb{C}[\underbrace{x_1^2 + x_2^2 + \dots + x_n^2}_{N^2}] \qquad N^2(v) = \|v\|^2$$

d) $G = GL_n$, $V = \mathrm{Mat}_n$ action is conjugation: $g \cdot X := gXg^{-1}$

Char poly:

$$ch_X(t) = \det(tI - X) = t^n + \sum_{i=1}^{n} (-1)^i e_i(X) t^{n-i}.$$

where $e_i := e_i(X)$ is the ith elem. sym. poly in the evalues of X.

Then conj. preserves $ch_X(t)$, so

$$\mathbb{C}[e_1, \dots, e_n] \subseteq \mathbb{C}[\mathrm{Mat}_n]^{GL_n}$$

On the other hand, if $f \in$ ⤴, then if $D = \{\mathrm{diag}(a_1, \dots, a_n)\}$ $f|_D$ is symm. in $a_1, \dots, a_n$, and a similar continuity argument to the above shows that $f \in \mathbb{C}[e_1, \dots, e_n]$

Also note that $\mathrm{Tr}_k : X \mapsto \mathrm{Tr}(X^k)$ is the kth power sum sym. fun. in the evalues of $X$, so

$$\mathbb{C}[\mathrm{Mat}_n]^{GL_n} \cong \mathbb{C}[\mathrm{Tr}_1, \mathrm{Tr}_2, \ldots, \mathrm{Tr}_n]$$

Other examples: [Kraft-Procesi]

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Questions:

Q1: Is $\mathbb{C}[V]^G$ finitely generated?

Q2: What is a set of generators?

First Fundamental Theorem (FFT)

Q3: What are the relations btwn. these generators?

Second Fundamental Theorem (SFT)

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Thm 83 (Hilbert's Finiteness Thm): If $G$ has complete reducibility (reductive), then $\mathbb{C}[V]^G$ is finitely generated.

(In general, ans. to Q1 is "no")

Def 84: A <u>Reynolds operator</u> is a linear map $R : \mathbb{C}[V] \to \mathbb{C}[V]^G$ satisfying

- $R(f) = f \quad \forall f \in \mathbb{C}[V]^G$
- $R(f \cdot \varphi) = R(f) \cdot \varphi \quad \forall f \in \mathbb{C}[V],\ \varphi \in \mathbb{C}[V]^G$

$R$ always exists if $G$ is red. (averaging op. if $G$: compact)

Pf sketch of Thm 83: Write $J = \mathbb{C}[V]^G$, $J_+ = \{f \in J \mid f(0) = 0\}$.
Let $I$ be the ideal $I = (J_+) \subseteq \mathbb{C}[V]$.

By the Hilbert basis thm, $\mathbb{C}[V]$ is noetherian, so $I$ is a finitely gen'd ideal, $I = (\phi_1, \dots, \phi_m)$, $\phi_i \in J_+$, homog.

We claim that $J = \mathbb{C}[\phi_1, \dots, \phi_m]$

Let $\phi \in J_+$ be homog. of deg $d > 0$, and induct on $d$.
Since $\phi \in J_+ \subseteq I$, write

$$\phi = \sum_i f_i \phi_i \quad \text{w/ } f_i \in \mathbb{C}[V].$$

Apply the Reynolds op:

$$\phi = R(\phi) = \sum_i R(f_i \phi_i) = \sum_i R(f_i)\,\phi_i.$$

$$J$$

We have $\deg R(f_i) = d - \deg \phi_i < d$, so by induction each $R(f_i) \in \mathbb{C}[\phi_1, \dots, \phi_m]$; hence so is $\phi$. $\square$