Extra topics: we'll try to cover all three: invariant theory
quiver reps
quantum groups

Today: Pf of

Thm 42 (Murnaghan–Nakayama rule): For all $\lambda, \mu \vdash n$,

$$\chi^\lambda(w_\mu) = \sum_{\xi} (-1)^{h(\xi)} \chi^{\lambda \setminus \xi}(w_{\mu'})$$

where the sum is over all border strips of λ of size μ_1.

Recall: Frobenius characteristic map $ch : R \to \Lambda$

$$ch(f) = \sum_{\mu \vdash k} \frac{1}{z_\mu} f(w_\mu) P_\mu ,$$

Thm 79:

a) $ch(\chi^\lambda) = s_\lambda$

b) ch is a bijective linear map

c) ch is an isometry w.r.t. the std. inner prod. on class functions and the inner prod. on Λ^k defined by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

d) ch is an algebra isomorphism w.r.t. mult. in Λ and the induction product

$$R^k \times R^m \to R^{k+m}$$
$$\chi \cdot \psi \mapsto \mathrm{Ind}_{S_k \times S_m}^{S_{k+m}} (\chi \otimes \psi)$$

Pf: a) Consider the S_k class function $w \mapsto Tr_{V^{\otimes k}}(g, w)$.

By Prop 77, $Tr_{V^{\otimes k}}(g,w) = P_\mu(x)$, where $x_1, \dots, x_n$ are the evalues of g.

On the other hand, by Schur-Weyl duality, if $n \geq k$,

$$Tr_{V^{\otimes k}}(g, w) = \sum_{\lambda \vdash k} \underbrace{Tr_{V^\lambda}(g)}_{s_\lambda(x)} \cdot \chi^\lambda(w).$$

Equating, and applying the Frobenius characteristic map gives

$$\sum_{\mu \vdash k} \frac{1}{z_\mu} P_\mu(x) P_\mu(y) = \sum_{\lambda \vdash n} s_\lambda(x) \, ch(\chi^\lambda)(y).$$

By the Cauchy identity and linear independence of the $s_\lambda(y)$, we have $ch(\chi^\lambda) = s_\lambda$.

b), c) Follow since the χ^λ and s_λ are orthonormal bases

d) Frobenius reciprocity (HW 5)

□

Cor 80 (Frobenius character formula):

$$s_\lambda = \sum_{\mu \vdash k} \frac{1}{z_\mu} \chi^\lambda(w_\mu) P_\mu,$$

So $\chi^\lambda(w_\mu)$ is z_μ times the coeff. of P_μ in the expansion of s_λ in the power sum basis.

Pf: Apply the defn of the Frobenius characteristic and Thm 79(a). □

Prop 81: For all μ and all k,

$$p_k s_\mu = \sum_\lambda (-1)^{h(\lambda/\mu)} s_\lambda$$

where $\lambda/\mu := \lambda \setminus \mu$ is a border-strip of size k.

Prop 81 $\Rightarrow$ Thm 42:

First we expand p_μ in the Schur basis.

Note that $ch(z_\mu \delta_{\mu\lambda}) = p_\mu$, so

$$\langle p_\mu, p_\nu \rangle \overset{\text{isom.}}{=} \langle z_\mu \delta_{\mu\lambda}, z_\nu \delta_{\nu\lambda} \rangle = \frac{1}{k!} \sum_{w \in C_\mu \cap C_\nu} z_\mu z_\nu$$

$$= z_\mu z_\nu \frac{|C_\mu|}{k!} \delta_{\mu\nu} = z_\mu \delta_{\mu\nu}.$$

By the Frobenius character formula,

$$\langle s_\lambda, p_\mu \rangle = \sum_\nu \frac{1}{z_\nu} \chi^\lambda(w_\nu) \langle p_\mu, p_\nu \rangle = \chi^\lambda(w_\mu),$$

so since the s_λ are orthonormal,

$$p_\mu = \sum_\nu \chi^\nu(w_\mu) s_\nu.$$

Now, let $\mu' = (\mu_2, \dots \mu_{\ell(\mu)})$. Then

$$\chi^\lambda(w_\mu) = \langle s_\lambda, p_{\mu_1} p_{\mu'} \rangle = \left\langle s_\lambda, p_{\mu_1} \sum_\nu \chi^\nu(w_{\mu'}) s_\nu \right\rangle$$

$$= \sum_{\nu} \chi^{\nu}(w_{\mu'}) \langle s_{\lambda}, p_{\mu_1} s_{\nu} \rangle$$

$$= \sum_{\nu} \chi^{\nu}(w_{\mu'}) \begin{cases} (-1)^{h(\lambda/\nu)}, & \text{if } \lambda/\nu \text{ is a border strip of size } \mu_1 \\ 0, & \text{else} \end{cases}$$

Prop 81

as desired.

$\square$

Pf of Prop 81: Let $n >> 0$. [Stanley, § 7.17]

Step 1:

$$a_{\alpha} p_k = \sum_{j=1}^{n} a_{\alpha + r\varepsilon_j} \quad \text{where } a_{\alpha} = \sum_{w \in S_n} (-1)^{w} x^{w(\alpha)}, \quad \varepsilon_j = (0, \dots, \underset{j}{1}, 0, \dots, 0)$$

Pf:

$$a_{\alpha} p_k = \sum_{w \in S_n} (-1)^{w} \sum_{i=1}^{n} x^{w(\alpha) + k\varepsilon_i}$$

$$= \sum_{w \in S_n} (-1)^{w} \sum_{j=1}^{n} x^{w(\alpha) + k\varepsilon_{w(j)}} \qquad j = w^{-1}(i)$$

$$= \sum_{w \in S_n} (-1)^{w} \sum_{j=1}^{n} x^{w(\alpha + k\varepsilon_j)}$$

$$= \sum_{j=1}^{n} a_{\alpha + k\varepsilon_j}.$$

Step 2: Expand $p_k s_\mu$:

By Step 1 and the bialternant formula,

$$p_k s_\mu = \frac{p_k a_{\mu+\rho}}{a_\rho} = \sum_{j=1}^{n} \frac{a_{\alpha+k\varepsilon_j}}{a_\rho} \qquad \alpha = \mu+\rho$$

If $\alpha_j + k = \alpha_i$ for some i, then $a_{\alpha+k\varepsilon_j} = 0$. Otherwise, there is a unique rearrangement $\nu := \nu^{(j)}$ of $\alpha+k\varepsilon_j$ that is a strict partition: $\nu_1 > \nu_2 > \cdots$. and

$$a_\nu = (-1)^{j-p} a_{\alpha+k\varepsilon_j}$$

where $\nu_p = (\alpha+k\varepsilon_j)_j$.

So
$$p_k s_\mu = \sum_j (-1)^{j-p} \frac{a_{\nu^{(j)}}}{a_\rho} = \sum_j (-1)^{j-p} s_{\underbrace{\nu^{(j)}-\rho}_{=:\lambda(j)}}$$

Step 3: The result now follows from the claim that

$\lambda^{(j)} = \mu$ + a/the border strip starting in row j of size k and $h(\lambda/\mu) = j-p$.

We "prove" this by example:

$\mu = (6, 4, 3, 3, 1)$ $\qquad j = 5, \; k = 4$

$\alpha = \mu + \rho = (10, 7, 5, 4, 1)$

$\alpha + \varepsilon_j = (10, 7, 5, 8, 1)$

$\nu = (10, 8, 7, 5, 1)$

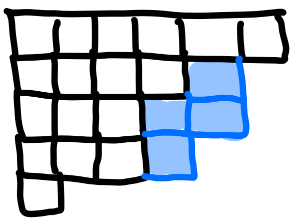
$\lambda = \nu - \rho = (6, 5, 5, 4, 1)$

$$\mu = (6, 4, 3, 3, 1)$$

$$\lambda = (6, 5, 5, 4, 1)$$

$p = 2 \qquad j - p = 2 = h(\lambda/\mu)$

μ



Starting next week: extra topics!