Last time: Schur-Weyl duality: As a $(GL_n \times S_k)$-repn,

$$V^{\otimes k} \cong \bigoplus_{\lambda \in Par(k,n)} V^\lambda \otimes S^\lambda.$$

Cor 75: In the highest wt. repn. V^λ, the wt. space multiplicities are given by the kostka numbers:

$$\dim \underbrace{V^\lambda_\mu}_{\substack{\text{wt. space for } \mu \\ \text{inside } V^\lambda}} = K_{\lambda\mu}.$$

Pf: Using Schur-Weyl duality, treating $V^{\otimes k}$ as an S_k-repn, we have

$$V^{\otimes k} \cong \bigoplus_{\lambda \in Par(k,n)} V^\lambda \otimes S^\lambda \cong \bigoplus_{\lambda,\mu} \left(V_\mu \cap (V^\lambda \otimes S^\lambda)\right)$$

$$\cong \bigoplus_{\lambda,\mu} \left(V^\lambda_\mu \otimes S^\lambda\right)$$

and on V_μ, using Lemma 73, this gives

$$\bigoplus_\lambda K_{\lambda\mu} S^\lambda \cong M^\mu \cong V_\mu \cong \bigoplus_\lambda \left(V^\lambda_\mu \otimes S^\lambda\right) \cong \bigoplus_\lambda \left(\dim V^\lambda_\mu\right) S^\lambda$$

$\square$

Symmetric functions [Macdonald], [Stanley]

Def 76: The ring of symmetric functions is the ring $\Lambda \subseteq \mathbb{C}[[x_1, x_2, \dots]]$ consisting of power series of bounded degree that are sym. under permatations of the x_i.

Setting $x_{n+1} = x_{n+2} = \cdots = 0$ gives sym. polys. in n vars.

Several important beses, all homog. of deg $|\lambda|$.

- elementary symmetric functions

$$e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k} \qquad e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots$$

- complete homogeneous symmetric functions

$$h_k = \sum_{i_1 \le \cdots \le i_k} x_{i_1} \cdots x_{i_k} \qquad h_\lambda = h_{\lambda_1} h_{\lambda_1} \cdots$$

- monomial symmetric functions

$$m_\lambda = \sum_{\substack{i_1, \dots, i_\ell \\ \text{distinct}}} x_{i_1}^{\lambda_1} \cdots x_{i_\ell}^{\lambda_\ell}$$

- power sum symmetric functions

$$p_k = \sum_{i \ge 1} x_i^k \qquad p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots$$

- Schur functions

$$s_\lambda = \sum_{\substack{\text{SSYT } T \\ \text{of shape} \\ \lambda}} x^{wt(T)} = \sum_{\mu:\text{comp.}} K_{\lambda\mu} x^\mu = \sum_{\substack{\mu:\text{part} \\ \mu \trianglelefteq \lambda}} K_{\lambda\mu} m_\mu$$

$s_{(k)} = h_k, \; s_{(1^k)} = e_k$

Schur functions satisfy several nice identities, including:
(see e.g. [Macdonald, Ch 1])

- Cauchy bialternant/ Weyl character:

$$s_\lambda(x_1,\dots,x_n) = \frac{\sum_{w\in S_n} (-1)^w x^{w(\lambda+\rho)}}{\sum_{w\in S_n} (-1)^w x^{w\rho}} \quad \begin{matrix} =: a_{\lambda+\rho} \\ \\ =: a_\rho = \prod_{i<j}(x_i - x_j) \end{matrix}$$

In particular, it is the character of the highest wt repn V^λ

- Cauchy identity:

$$\prod_{i,j} \frac{1}{1-x_i y_j} = \sum_\lambda s_\lambda(x) s_\lambda(y) = \sum_\mu \frac{1}{z_\mu} p_\mu(x) p_\mu(y)$$

where $z_\mu = \mu_1 \mu_2 \cdots \, m_1(\mu)! \underbrace{m_2(\mu)!}_{\text{num parts of size 2}} \cdots$

- Pieri rule:

$$h_k s_\mu = \sum_\lambda s_\lambda, \quad e_k s_{\mu'} = \sum_\lambda s_{\lambda'}$$

where the sums are over all λ s.t. $|\lambda| = |\mu| + k$ and
$\lambda'_i - \mu'_i \leq 1 \ \forall i$

- Jacobi-Trudi rule:

$$s_\lambda = \det\left[h_{\lambda_i + j - i}\right]_{1\leq i,j\leq \ell(\lambda)} = \det\left[e_{\lambda'_i + j - i}\right]_{1\leq i,j \leq \ell(\lambda')}$$

Prop 77: Let $g \in GL_n$ have evalues $x_1, \ldots, x_n$, and let $w \in S_k$ have cycle type $\mu = (\mu_1, \ldots, \mu_\ell)$. Then,

$$\mathrm{Tr}_{V^{\otimes k}}(g, w) = p_\mu(x_1, \ldots, x_n)$$

Pf: By continuity and conjugation, we can assume $g = \mathrm{diag}(x_1, \ldots, x_n)$. Then,

$$(g, w) \cdot e_I = x_{i_1} \cdots x_{i_k} e_{w \cdot I}$$

where $w \cdot I = (i_{w^{-1}(1)}, \ldots, i_{w^{-1}(k)})$.

Now, $e_{w \cdot I} = e_I$ iff i_j is constant on each cycle of w. For such an I, let $j_1, \ldots, j_\ell$ denote the values of i on each conj. class.

Thus we have

$$\mathrm{Tr}_{V^{\otimes k}}(g, w) = \sum_{\substack{I \\ w \cdot I = I}} x_{i_1} \cdots x_{i_k}$$

$$= \sum_{1 \leq j_1, \ldots, j_\ell \leq n} x_{j_1}^{\mu_1} \cdots x_{j_\ell}^{\mu_\ell}$$

$$= p_{\mu_1}(x) p_{\mu_2}(x) \cdots p_{\mu_\ell}(x) \quad \text{where } x = (x_1, \ldots, x_n)$$

$$= p_\mu(x).$$

$\square$

Let R^k be the space of class functions on S_k, $R = \bigoplus_{k\geq 0} R^k$

Let Λ^k be the space of homog. sym. funs. of deg k.

Recall: χ^λ: character of S^λ, w_μ: perm. of cycle type μ

Def 78: The Frobenius characteristic map $ch: R \to \Lambda$ is the map

$$ch(f) = \sum_{\mu \vdash k} \frac{1}{z_\mu} f_\mu p_\mu,$$

where f_μ is the value of f on w_μ

Thm 79:

a) $ch(\chi^\lambda) = s_\lambda$

b) ch is a bijective linear map

c) ch is an isometry w.r.t. the std. inner prod. on class functions and the inner prod. on Λ^k defined by $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

d) ch is an algebra isomorphism w.r.t. mult. in Λ and the induction product

$$R^k \times R^m \to R^{k+m}$$
$$\chi \cdot \psi \mapsto \mathrm{Ind}_{S_k \times S_m}^{S_{k+m}} (\chi \otimes \psi)$$

Pf: a) Consider the S_k class function $w \mapsto \mathrm{Tr}_{V^{\otimes k}}(g,w)$.

By Prop 77, $\mathrm{Tr}_{V^{\otimes k}}(g,w) = p_\mu(x)$, where $x_1,\dots,x_n$ are the evalues of g.

On the other hand, by Schur-Weyl duality, if $n \geq k$,

$$\mathrm{Tr}_{V^{\otimes k}}(g,w) = \sum_{\lambda \vdash k} \underbrace{\mathrm{Tr}_{V^\lambda}(g)}_{s_\lambda(x)} \cdot \chi^\lambda(w).$$

Equating, and applying the Frobenius characteristic map gives

$$\sum_{\mu \vdash k} \frac{1}{z_\mu} p_\mu(x) p_\mu(y) = \sum_{\lambda \vdash n} s_\lambda(x)\, \mathrm{ch}(\chi^\lambda)(y).$$

By the Cauchy identity and linear independence of the $s_\lambda(y)$, we have $\mathrm{ch}(\chi^\lambda) = s_\lambda$.

b), c) Follow since the χ^λ and s_λ are orthonormal bases

d) Frobenius reciprocity (HW 5?)

$\square$

Cor 80 (Frobenius character formula):

$$s_\lambda = \sum_{\mu \vdash k} \frac{1}{z_\mu} \chi^\lambda(w_\mu)\, p_\mu,$$

so $z_\mu^{-1} \chi^\lambda(w_\mu)$ is the coeff. of p_μ in the expansion of s_λ in the power sum basis.

Pf: Apply the defn of the Frobenius characteristic and Thm 79(a). $\square$