Last time: Using the double centraliser thm (Thm 71),

Prop 72: V std. repn of GL_n,

$A = \rho(GL_n)$, $B = \pi(S_k) \subseteq \mathrm{End}(V^{\otimes k})$

We have $\mathrm{Comm}(A) = B$ and $\mathrm{Comm}(B) = A$.

As a $(GL_n \times S_k)$-repn,

$$(*) \qquad V^{\otimes k} \cong \bigoplus_i E^i \otimes F^i,$$

where the E^i are distinct GL_n-irreps.

and the F^i are distinct S_k-irreps.

Today: Which repns are these?

Let $T \cong (\mathbb{C}^\times)^n \leq GL_n$ be the subgp. of diag. matrices.

Since T is abelian, by HW4 #1 $V^{\otimes k}$ is a direct sum of T wt. spaces:

$$V^{\otimes k} = \bigoplus_{\mu : wt} V_\mu \qquad \mathrm{diag}(z_1, \dots, z_n) v = \mu(z_1, \dots, z_n) v = z_1^{\mu_1} \cdots z_n^{\mu_n} v \quad \forall v \in V_\mu$$

μ is a composition $(\mu_1, \dots, \mu_n)$, $\mu_1 + \dots + \mu_n = k$

In fact,

$V_\mu = \mathrm{span}\{ e_I \mid \mu_I = \mu \}$ where $(\mu_I)_j = \# e_j$ appearing in e_I

So $V_\mu \neq 0 \iff \mu \in \mathrm{Comp}(k,n)$ compositions of k w/ n parts

In particular (using Thm. 70), every E^i is a <u>polynomial</u> repn. of GL_n.

Since the actions of T and S_k commute, V_μ is an S_k-module $\forall \mu \in \mathrm{Comp}(k,n)$.

Basis vectors $e_I \in V_\mu$

$$I = (i_1, \dots, i_n), \qquad \mu_j \text{ copies of } j$$

are in (S_k-equiv.) bij. w/ diagrams

$$\begin{array}{llll} a_1 & a_2 & \cdots & \leftarrow \text{positions of 1's in } I \text{, in inc. order} \\ b_1 & b_2 & \cdots & \leftarrow \quad \text{"} \quad \text{2's} \quad \text{"} \\ c_1 & c_2 & \cdots & \leftarrow \quad \text{"} \quad \text{3's} \quad \text{"} \end{array}$$

Tabloids!

Thus, we have proved:

Lemma 73: As an S_k-module,

$$V_\mu \cong M^\mu,$$

where M^μ is the span of μ-tabloids w/ action as in Def. 24.

Note that:

$$M^\mu \cong M^\lambda \qquad \text{for any rearrangement } \lambda \text{ of the parts of } \mu$$

Now, since every S^λ appears in some M^μ and by Thm 40, we have $|\lambda| = |\mu| = k$, $\ell(\lambda) \leq \ell(\mu) = n$, and $\lambda \trianglerighteq \mu$.

Since the V_μ span $V^{\otimes k}$ and since the F^i in (*) are distinct S_k-irreps, we can rewrite (*):

$$V^{\otimes k} \cong \bigoplus_{\lambda \in Par(k,n)} E^\lambda \otimes S^\lambda,$$

where $Par(k,n)$ is the set of partitions of k w/ length $\leq n$.

It remains to determine the E^λ. Fix $\lambda \in Par(k,n)$.
Let v be a GL_n-highest-wt. vector in $E^\lambda \otimes S^\lambda$, and let μ be its weight, so that $v \in V_\mu \cong M^\mu$.
We will show that μ can be taken to equal λ, and by uniqueness, it must be λ.
As in the pf. of Thm. 40, M^μ can be identified w/ $\mathbb{C}[\mathcal{T}_{\lambda\mu}]$, the span of tableaux of shape λ, content μ.
Furthermore, $v \in S^\lambda \subseteq M^\mu$ is a linear comb. of elts.

$$v = \sum_{\omega, T} c_{\omega,T} \, v_{\omega,T}, \qquad v_{\omega,T} := \kappa_{t_\lambda} \sum_{S \in \{T\}} \omega S.$$

Unless $\lambda \trianglerighteq \mu$, $v_{T,T'} = 0 \ \forall \, T, T' \in \mathbb{C}[\mathcal{T}_{\lambda\mu}]$. $\quad t_\lambda = \begin{array}{l} 1\ 2 \cdots \\ \lambda_1{+}1 \cdots \\ \cdots \end{array}$

Since v is highest-wt., it is highest wt. for the corresponding $\mathfrak{gl}_n$-repn., so the raising operators

$$X_i = E_{i,i+1} = \begin{array}{cc} \begin{bmatrix} 0 & \ddots & 1 \\ & & 0 \end{bmatrix} & i \\ \quad\quad i+1 & \end{array}$$

usually called e_i, but changed for notation reasons

have $X_i v = 0 \ \forall i$. Here the action is

$$X_i(S) = \sum S'$$

where $S' \in R_i(S) := \{$tableaux formed from S by changing an $i+1$ to an $i\}$

We have $S \in \mathcal{T}_{\lambda\mu} \Rightarrow S' \in \mathcal{T}_{\lambda\nu}$ where

$$\nu = (\mu_1, \ldots, \mu_{i-1}, \mu_i + 1, \mu_{i+1} - 1, \mu_{i+2}, \ldots, \mu_n) \trianglerighteq \mu.$$

Thus,

$$X_i(V_{w,T}) = K_{t_\lambda} \sum_{S \in \{T\}} w\, X_i(S) = V_{w, X_i(T)} \in S^\lambda \cap M^\nu$$

So taking $\lambda = \mu$ gives $V_{w, X_i(T)}$ $\forall i$, so v is highest wt.

We have proved:

Thm 74 (Schur-Weyl duality): As a $(GL_n \times S_k)$-repn,

$$V^{\otimes k} \cong \bigoplus_{\lambda \in Par(k,n)} V^\lambda \otimes S^\lambda.$$

If $n \geq k$, then all irreps. of S_k appear.

Cor 75: In the highest wt. repn. V^λ, the wt. space multiplicities are given by the kostka numbers:

$$\dim \underbrace{V_\mu^\lambda}_{\substack{\text{wt. space for } \mu \\ \text{inside } V^\lambda}} = K_{\lambda\mu}.$$

Pf: Combining Lemma 73 and Thm. 74, treating $V^{\otimes k}$ as an S_k-repn, we have

$$V^{\otimes k} \cong \bigoplus_{\lambda \in Par(k,n)} V^\lambda \otimes S^\lambda \cong \bigoplus_{\substack{\lambda,\mu \\ \lambda \trianglerighteq \mu}} (V_\mu \cap (V^\lambda \otimes S^\lambda))$$

$$\cong \bigoplus_{\substack{\lambda,\mu \\ \lambda \trianglerighteq \mu}} (V_\mu^\lambda \otimes S^\lambda))$$

and on V_μ, this gives

$$\bigoplus_\lambda K_{\lambda\mu} S^\lambda \cong M^\mu = V_\mu \cong \bigoplus_\lambda (V_\mu^\lambda \otimes S^\lambda)) \cong \bigoplus_\lambda (\dim V_\mu^\lambda) S^\lambda$$

$\square$

Symmetric functions

Def 76: The ring of symmetric functions is the ring $\Lambda = \mathbb{C}[x_1, x_2, \ldots]^{S_n}$, where the S_n action permutes the variables.

Several important families:

- elementary symmetric functions (alg. indep. gens.)

$$e_k = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k}$$

- complete homogeneous symmetric functions (alg. indep. gens.)

$$h_k = \sum_{i_1 \leq \cdots \leq i_k} x_{i_1} \cdots x_{i_k}$$

- monomial symmetric functions (basis)

$$m_\lambda = \sum_{\substack{i_1, \ldots, i_\ell \\ \text{distinct}}} x_{i_1}^{\lambda_1} \cdots x_{i_\ell}^{\lambda_\ell}$$

- power sum symmetric functions (alg. indep. gens.)

$$p_n = \sum_{i \geq 1} x_i^n$$

- Schur functions (basis)

$$s_\lambda = \sum_{\substack{\text{SSYT } T \\ \text{of shape} \\ \lambda}} x^{wt(T)} = \sum_{\mu:\text{comp.}} K_{\lambda\mu} x^\mu = \sum_{\substack{\mu:\text{part} \\ \mu \trianglelefteq \lambda}} K_{\lambda\mu} m_\mu$$

$$s_{(k)} = h_k, \quad s_{(1^k)} = e_k$$