Announcement

New classroom : Noyes Lab. 164

Last time: Basic Def'ns and examples

Today: Schur's Lemma & Maschke's Theorem [FH, §1-2]

Def 3: A repn V is <u>completely reducible</u> if it is (isom to) a direct sum of irreps.:

multiplicities

$$V = V_1 \oplus \cdots \oplus V_m = \underbrace{c_1 V_1}_{\underbrace{V_1 \oplus \cdots \oplus V_1}_{c_1}} \oplus \cdots \oplus c_k V_k$$

mutually nonisom.

Recall Ex. 3 from last time was an example of a repn that was not completely reducible

---

Now for the next several weeks let us restrict to the case where G is a finite gp. and V is a f.d. complex v.s.

Prop 4 (Schur's Lemma): Let V, W be G-irreps. and $\varphi \in \mathrm{Hom}_G(V, W)$.

a) Either φ is an isom. or $\varphi = 0$.

b) If $V = W$, then φ is a scalar mult. of the identity

Pf:

a) $\ker \varphi$ and $\mathrm{im}\, \varphi$ are subrepns of V and W. Since V & W are irreps, we must have either $\underbrace{\ker \varphi = 0,\ \mathrm{im}\, \varphi = W}_{\text{isom.}}$ or $\underbrace{\ker \varphi = V,\ \mathrm{im}\, \varphi = 0}_{\varphi = 0}$.

b) Since $\mathbb{C}$ is alg. closed, φ must have an e-value λ. This means that $\ker(\varphi - \lambda I) \neq 0$, so by a), $\varphi - \lambda I$ is the zero map, and $\varphi = \lambda I$.

$\square$

Thm 5 (Maschke's Thm): Every finite dimensional complex repn. V of a finite gp. is completely reducible. Moreover, the decomposition

$$V = c_1 V_1 \oplus \cdots \oplus c_k V_k$$

is unique up to isomorphism and the order of the summands

Pf: If V is irred., we're done, so suppose V has a proper nontrivial subrepn W. We want to find a complementary G-invariant subspace i.e. a subrepn U s.t. $V = W \oplus U$.

To do so, we use "Weyl's averaging trick" to construct a G-invariant inner product.

Let $\langle \cdot , \cdot \rangle$ denote any Hermitian inner product on V (e.g. $\langle v, w \rangle = v^T \bar{w}$).

Define

$$\langle v, w \rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle$$

Now, $\langle \cdot , \cdot \rangle_G$ is also a Hermitian inner product

- conjugate symmetric
- linear in first argument
- positive definite

We claim that $\langle \cdot , \cdot \rangle_G$ is G-invariant:

$$\langle gv, gw \rangle_G = \langle v, w \rangle_G \quad \forall g \in G$$

Indeed,

$$\langle gv, gw\rangle_G = \frac{1}{|G|}\sum_{h\in G}\langle hgv, hgw\rangle$$

$$= \frac{1}{|G|}\sum_{h'\in G}\langle h'v, h'w\rangle = \langle v, w\rangle_G.$$

Let $U = W^\perp$, the orthogonal complement w.r.t. $\langle\cdot,\cdot\rangle_G$. Then $V = U \oplus W$ as v.s., so we only need to show that U is G-invariant. But if $u \in U$, then for all $w \in W$, $g \in G$,

$$\langle gu, w\rangle_G = \langle u, \underbrace{g^{-1}w}_{\in W}\rangle_G = 0,$$

so $gu \in W$, and W is a subrepn.

Now, since G is f.d., we use induction on dimension and write U, W as a direct sum of irreps., making V a direct sum of irreps. too.

Finally, suppose

$$V \cong c_1V_1 \oplus \cdots c_kV_k \cong d_1W_1 \oplus \cdots \oplus d_\ell W_\ell$$

are two irred. decomps. of V. Write the identity map in block matrix form

$$\begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1k} \\ \vdots & & \vdots \\ \varphi_{\ell 1} & \cdots & \varphi_{\ell k} \end{bmatrix}$$

where $\varphi_{ij} : c_i V_i \mapsto d_j W_j$

By Schur's Lemma, $\varphi_{ij} = 0$ unless $V_i \cong W_j$, and then we must also have $c_i = d_j$ and φ_{ij} is an isomorphism $c_i V_i \to d_j W_j$. Thus, $k = \ell$ also and the two decomps. are equiv. $\square$

Remark: This argument works over most fields, but fails over $\mathbb{F}_p$ when $p \mid G$. Recall the example

$$G = \mathbb{Z}/p\mathbb{Z} = \langle g \rangle, \quad V = \mathbb{F}_p^2 \quad g^a \mapsto \begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}$$

$W = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$: invariant subspace

Let $\langle \cdot , \cdot \rangle$ be the standard inner prod. on $\mathbb{F}_2$ and let

$$\langle v, w \rangle_G = \sum_{g \in G} \langle gv, gw \rangle$$

Then

$$\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle_G = \sum_{a \in \mathbb{F}_p} \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle$$

$$= p \cdot 1 = 0$$

Which would imply that $W \subseteq W^{\perp}$

i.e. $\langle \cdot , \cdot \rangle_G$ is not an inner product

Remark: Note that the uniqueness only holds up to isom.

e.g. $V = V_{triv} \oplus V_{triv}$

$$V = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle \oplus \left\langle \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle \oplus \left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\rangle \oplus \left\langle \begin{bmatrix} -2 \\ 5 \end{bmatrix} \right\rangle$$

all valid irred. decomps.