Recall:

Irreps of S_k: Specht modules S^λ, $\lambda \vdash k$

Poly irreps of $GL_n(\mathbb{C})$: highest wt. repns V^λ, $\underbrace{\ell(\lambda) \leq n}_{\text{remove trailing zeroes}}$

Today: duality btwn. these two theories

First, to sum up last time:

Def/Thm 70: A repn. π of $GL_n(\mathbb{C})$ is <u>polynomial</u> if all its matrix coefficients $g \mapsto \langle e_j, \pi(g) e_i \rangle$ are polynomial funs. (in the entries) of g. π is <u>rational</u> if its matrix coeffs. are rational funs. of g.
If $\lambda = (\lambda_1, \dots, \lambda_n)$, $\lambda_1 \geq \dots \geq \lambda_n$ is a dominant wt., then the (exponentiated) highest wt. repn. V^λ is rational, and if $\lambda_n \geq 0$, it is polynomial. Every rational repn can be written in the form $\det^{\otimes(-a)} \otimes V^\lambda$ where V^λ is poly.

Let V be a f.d. v.s. If $A \subseteq \operatorname{End} V$, let

$$\operatorname{Comm}(A) = \{ b \in \operatorname{End} V \mid ab = ba \ \forall a \in A \}$$

Thm 71 (Double commutant/centralizer theorem):

Let $A \subseteq \mathrm{End}(V)$ be a subalgebra s.t. V is a completely reducible A-module. Let $B = \mathrm{Comm}(A)$.

a) $\mathrm{Comm}(B) = A$.

b) V is completely reducible as a B-module.

c) As an $(A \otimes B)$-module, we have the decomposition:

$$V \cong \bigoplus_i U_i \otimes W_i,$$

where U_i (resp. W_i) are the distinct A-irreps. (resp. B-irreps) appearing in V.

U_i and W_i determine each other, and

$$\mathrm{Hom}(U_i, V) \cong W_i \quad \text{as } B\text{-modules}$$

$$\mathrm{Hom}(W_i, V) \cong U_i \quad \text{as } A\text{-modules}$$

Pf: See [Goodman-Wallach Thms. 4.1.13, 4.2.1]. $\square$

Set $GL_n = GL_n(\mathbb{C})$, $V = \mathbb{C}^n$ the std. repn.

$$V^{\otimes k} = \operatorname{span}\{v_1 \otimes \cdots \otimes v_k \mid v_i \in V\}$$

is a GL_n-repn ρ under

$$g \cdot (v_1 \otimes \cdots \otimes v_k) = gv_1 \otimes \cdots \otimes gv_k$$

and an S_k-repn π under

$$w \cdot (v_1 \otimes \cdots \otimes v_k) = v_{w^{-1}(1)} \otimes \cdots \otimes v_{w^{-1}(k)}.$$

Let $A = \operatorname{span}(\rho(GL_n)) \subseteq \operatorname{End}(V)$

$B = \operatorname{span}(\pi(S_k)) \subseteq \operatorname{End}(V)$

Prop 72: We have $\operatorname{Comm}(A) = B$ and $\operatorname{Comm}(B) = A$.

As a $(GL_n \times S_k)$-repn,

$$V^{\otimes k} \cong \bigoplus_i E^i \otimes F^i,$$

where the E^i are distinct GL_n-irreps.

and the F^i are distinct S_k-irreps.

(later, we will see which ones)

Pf: Since V is f.d. and since GL_n and S_k have complete reducibility, Thm. 71 applies, and if we can show that $\mathrm{Comm}(A) = B$ the rest of the statements follow.

By inspection, $B \subseteq \mathrm{Comm}(A)$ and $A \subseteq \mathrm{Comm}(B)$. Thus it suffices to show that $\mathrm{Comm}(B) \subseteq A$.

Let $\{e_1, \dots, e_n\}$ be the std. basis of V. For $I = (i_1, \dots, i_k)$, let $|I| = k$ and $e_I = e_{i_1} \otimes \cdots \otimes e_{i_k}$. Then $\{e_I \mid |I| = k\}$ is a basis for $V^{\otimes k}$, and the S_k action can be written

$$w e_I = e_{wI} \quad \text{where} \quad w \cdot I = (i_{w^{-1}(1)}, \dots, i_{w^{-1}(k)})$$

Write $T \in \mathrm{End}(V^{\otimes k})$ as a matrix relative to this basis:

$$T e_J = \sum_I a_{I,J} \, e_I$$

Then

$$T(w \cdot e_J) = T(e_{wJ}) = \sum_I a_{I,wJ} \, e_I$$

and

$$w \cdot T(e_J) = \sum_I a_{I,J} \, e_{wI} = \sum_I a_{w^{-1}I, J} \, e_I,$$

So $T \in \mathrm{Comm}(B) \iff a_{I,wJ} = a_{w^{-1}I,J} \quad \forall w, I, J$

(*) $\iff a_{wI,wJ} = a_{I,J} \quad \forall w, I, J.$

Consider the bilinear form

$$(X,Y) := \mathrm{tr}(XY) \text{ on } \mathrm{End}(V^{\otimes k})$$

This is nondegenerate i.e. $(X,Y)=0 \ \forall X \Rightarrow Y=0$.

We can project onto $\mathrm{Comm}(B)$ by averaging:

$$X \mapsto X^{\natural} = \frac{1}{k!} \sum_{w \in S_k} \pi(w) X \pi(w)^{-1} \in \mathrm{Comm}(B)$$

and if $T \in \mathrm{Comm}(B)$, then

$$(X^{\natural}, T) = \frac{1}{k!} \sum_{w \in S_k} \mathrm{tr}\left(\pi(w) X \pi(w)^{-1} T\right) = (X, T)$$

since trace is cyclicly invariant and T commutes w/ π.
Thus if $(X,T)=0 \ \forall X \in \mathrm{Comm}(B)$, then $(X,T)=0 \ \forall X \in \mathrm{End}(V^{\otimes k})$,
so since $(\cdot,\cdot)$ is nondegen, $T=0$. This means $(\cdot,\cdot)$ is nondegen on $\mathrm{Comm}(B)$.

To show that $\mathrm{Comm}(B) = A$, it thus suffices to show that if $T \in \mathrm{Comm}(B)$ satisfies $(X,T)=0 \ \underbrace{\forall X \in A}_{\text{i.e. } \forall X \in GL_n}$, then $T=0$.

If $g = [g_{ij}] \in GL_n$, then $\rho(g) = [g_{I,J}]$ where $I = (i_1, \dots, i_k)$, $J = (j_1, \dots, j_k)$,

$g_{I,J} := g_{i_1,j_1} \cdots g_{i_k,j_k}$. Now, we assume

$$(**) \quad 0 = (T, \rho(g)) = \sum_{I,J} a_{I,J}\, g_{J,I} \quad \forall g \in GL_n.$$

By continuity, $(**)$ also holds $\forall X = [x_{ij}] \in \mathrm{Mat}_n(\mathbb{C})$.

By $(*)$, if $X \in \mathrm{Mat}_n(\mathbb{C}) \subseteq \mathrm{Comm}(B)$, then

$X_{J,I} = X_{wJ,wI} \ \forall w \in W$

So we have

$$0 = \sum_{I,J} a_{I,J}\, x_{J,I} = \sum_{\gamma} n_\gamma a_\gamma x_\gamma,$$

where the sum is over a set of reps $\gamma = (I,J)$ of the S_k-orbits of the set of ordered pairs (I,J); $a_\gamma := a_{I,J}$ and $x_\gamma := x_{J,I}$ are constant on these orbits, and we set $n_\gamma := |S_k \cdot \gamma|$.

Think of this as a polynomial function in the n^2 variables $[x_{ij}]$. Then the x_γ are linearly independent, so $a_\gamma = 0 \ \forall \gamma$ and therefore $T = 0$. Hence, $A = \mathrm{Comm}(B)$. $\square$