Last two lectures: repn theory of compact gps.
In particular, their f.d. repns. are irred.

Def 68: Let G be a complex reductive gp.

a) G is semisimple if its Lie algebra is semisimple
b) G is reductive if its Lie algebra is reductive

(several variants of these defn's)

All of the complex Lie gps. from lecture 24 are reductive

Thm 69 (Weyl's Thm): All f.d. repns. of complex conn. semisimple Lie algebras and Lie gps. are completely reducible.

Will say that such an object has complete reduciblity

Very rough proof sketch: (see [Hall §6.1])

a) Compact gps. have complete reducibility (Thm. 64)

b) So Lie algebras of simply conn. compact Lie gps. have complete reducibility using the Lie gp.-Lie alg. correspondence (Prop. 61)

c) Every complex semisimple Lie algebra is the "complexification" of a Lie algebra from part b). Furthermore, complexification preserves complete reducibility

d) So again using the Lie gp.-Lie alg. correspondence (Prop. 61), all complex conn. semisimple Lie gps. have complete reducibility.

$\square$

What about the reductive case?

Ex: $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{z}$

where $\mathfrak{z} = \left\{ \begin{bmatrix} a & \\ & a \end{bmatrix} \middle| \, a \in \mathbb{C} \right\}$

$\pi : \begin{bmatrix} a & \\ & a \end{bmatrix} \mapsto \begin{bmatrix} & a \\ & \end{bmatrix}$ is a Lie alg. repn.

but $\begin{bmatrix} & a \\ & \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ay \\ 0 \end{bmatrix}$ only evector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So neither $\mathfrak{z}$ nor $\mathfrak{gl}_n(\mathbb{C})$ have complete reducibility!

Let's try to exponentiate this repn. using Prop. 61

$$e^{\begin{bmatrix} a & \\ & a \end{bmatrix}} = \begin{bmatrix} e^a & \\ & e^a \end{bmatrix}$$

$$\Pi\left(\begin{bmatrix} e^a & \\ & e^a \end{bmatrix}\right) = e^{\pi\left(\begin{bmatrix} a & \\ & a \end{bmatrix}\right)} = e^{\begin{bmatrix} & a \\ & \end{bmatrix}} = \begin{bmatrix} 1 & a \\ & 1 \end{bmatrix}$$

i.e.

$$\Pi\left(\begin{bmatrix} b & \\ & b \end{bmatrix}\right) = \begin{bmatrix} 1 & \log b \\ & 1 \end{bmatrix}$$

not a (continuous) repn.!

Note that $GL_n(\mathbb{C})$ (and $\mathbb{C}^\times = GL_1(\mathbb{C})$) are not simply connected

It turns out that $GL_n(\mathbb{C})$ <u>does</u> have complete reducibility:

$$GL_n(\mathbb{C}) = (\mathbb{C}^\times \times SL_n(\mathbb{C}))/\underbrace{\mu_n}_{\text{nth roots of } 1}$$

Thus every $GL_n(\mathbb{C})$ repn can be inflated to a $\mathbb{C}^\times \times SL_n(\mathbb{C})$ repn via

$$\mathbb{C}^\times \times SL_n(\mathbb{C}) \twoheadrightarrow GL_n(\mathbb{C}) \to V$$

The (holomorphic) irreps of $\mathbb{C}^\times$ are

$$z \mapsto z^n$$

The irreps of $SL_n(\mathbb{C})$ are the highest wt. repns. (exponentiated from the highest wt. repns. of $\mathfrak{sl}_n(\mathbb{C})$.

So the $GL_n(\mathbb{C})$ irreps turn out to be of the form

$g \mapsto \det(g)$

$$\underbrace{\det}_{\substack{\mathbb{C}^\times\text{-repn}\\ \text{"central character"}}}{}^{\otimes N} \otimes \underbrace{V}_{\substack{\text{highest}\\ \text{wt. repn}}}{}^{\lambda} \leftarrow \text{dominant wt.}$$

and all (f.d., holomorphic) $GL_n(\mathbb{C})$-repns are direct sums of these repns.

Now, dominant weights in type A_{n-1} are of the form

$$\lambda = a_1 e_1 + a_2(e_1+e_2) + \cdots + a_{n-1}(e_1 + \cdots + e_{n-1})$$
$$= (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \quad \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0, \quad \lambda_i \in \mathbb{Z}$$

Integer partitions w/ $\leq n-1$ parts!

Remember that we view e_i as the linear functional

$$e_i \begin{bmatrix} \ddots & & \\ & a_i & \\ & & \ddots \end{bmatrix} = a_i$$

In $\mathfrak{sl}_n(\mathbb{C})$, $e_n = -e_1 - \cdots - e_{n-1}$

But in $\mathfrak{gl}_n(\mathbb{C})$, it is linearly indep. and we have the weight

$(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{Z}$

Central character:

$$\lambda\left(\begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix}\right) = a(\lambda_1 + \cdots + \lambda_n)$$

Tensor by det $\leftrightarrow$ add 1 to each λ_i.

So, dominant means $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n$

Tensoring by $\det^{\otimes(-\lambda_n)}$ gives back the $\mathfrak{sl}$ condition

$$\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0$$

Exponentiating yields

$$\begin{array}{ccc} \mathrm{diag}(a_1, \ldots, a_n) & \longmapsto & a_1\lambda_1 + \cdots + a_n\lambda_n \\ \mathrm{exp} \downarrow & & \downarrow \mathrm{exp} \\ \mathrm{diag}(b_1, \ldots, b_n) & \longmapsto & b_1^{\lambda_1} \cdot \cdots b_n^{\lambda_n} \end{array}$$

and further arguments show that all the matrix coeffs. are Laurent polys. in the matrix entries

Def/Thm 70: A repn. π of $GL_n(\mathbb{C})$ is <u>polynomial</u> if all its matrix coefficients $g \mapsto \langle e_j, \pi(g) e_i \rangle$ are polynomial fns. (in the entries) of g. π is <u>rational</u> if its matrix coeffs. are rational fns. of g.

If $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \cdots \geq \lambda_n$ is a dominant wt., then the (exponentiated) highest wt. repn. V^λ is rational, and if $\lambda_n \geq 0$, it is polynomial. Every rational repn can be written in the form $\det^{\otimes(-a)} \otimes V^\lambda$ where V^λ is poly.