Announcement:
HW4 posted (due Mon. 4/20)

Thm 67 (Peter-Weyl Theorem): Let G be a compact group.

a) The matrix coeffs. of G for f.d. repns are dense in the set

$$C(G) = \{ f: G \to \mathbb{C} \mid f \text{ cont.} \}$$

of cont. fns. on G,

w.r.t. the L^∞-norm $|f - f'|_\infty = \sup_{g \in G} |f(g) - f'(g)|$

(and therefore in $L^2(G)$ since $C(G)$ is dense in $L^2(G)$).

b) Any unitary repn. of a compact gp. on a complex Hilbert space is a direct sum of finite dimensional irreducible representations.

c) We have the (completed) direct sum decomposition

$$L^2(G) = \widehat{\bigoplus_{\pi}} V_\pi^{\oplus \dim V_\pi}$$

where the sum is over all unitary irreps. (π, V_π)

We'll use convolution of functions:

$$(f * f')(g) := \int_G f(gh^{-1}) f'(h)\, dh = \int_G f(h) f'(h^{-1}g)\, dh$$

which is always defined if $f, f' \in C(G)$.

If $\phi \in C(G)$, let T_ϕ be the $L^2(G)$-operator

$$(T_\phi f) = \phi * f$$

Let $V(\lambda) := V(\lambda, \phi) = \{ f \in L^2(G) \mid T_\phi f = \lambda f \}$ be the λ eigenspace.

T_ϕ is bounded, satisfying $|T_\phi f|_\infty \leq |\phi|_\infty |f|_1 \; \forall f \in L^1(G)$ and self-adjoint if $\phi(g^{-1}) = \overline{\phi(g)}$.
Using the Spectral Thm. for compact operators, the $V(\lambda)$ are orthog., span $L^2(G)$, and if $\lambda \neq 0$, $V(\lambda)$ is f.d. In fact, if $\varepsilon > 0$, $\sum_{|\lambda| > \varepsilon} \dim V(\lambda) < \infty$

Bump Ch. 3–4

In addition it is inv. under right mult. $(\rho(g) \cdot f(h) = f(hg))$ since if $T_\phi f = \lambda f$,

$$(T_\phi \rho(g) f)(x) = \int_G \phi(xh^{-1}) f(hg)\, dh = \int_G \phi(xgh^{-1}) f(h)\, dh =$$

$$\rho(g)(T_\phi f)(x) = (\lambda \rho(g) f)(x)$$

Pf: a) Let $f \in C(G)$, $\varepsilon > 0$. We will show that there exists a matrix coeff. f' w/ $|f - f'|_\infty = 0$.

Since f is cont. on a compact set, $\exists$ a nbhd U of $1 \in G$ s.t.

$$|g \cdot f - f|_\infty < \varepsilon/2 \quad \forall g \in U$$

Let ϕ be a nonneg. fun supported on U s.t.

$\int_G \phi(g)\,dg = 1$ and $\phi(g^{-1}) = \overline{\phi(g)}$.

Then if $h \in G$,

$$\begin{aligned} |T_\phi f(h) - f(h)| &= \left| \int_G \left(\phi(g) f(g^{-1}h) - \phi(g) f(h)\right) dg \right| \\ &\leq \int_U \phi(g) \, |f(g^{-1}h) - f(h)| \, dg \\ &\leq \int_U \phi(g) \, |g \cdot f - f|_\infty \, dg \\ &\leq \int_U \phi(g) \cdot \tfrac{\varepsilon}{2} \cdot dg = \tfrac{\varepsilon}{2}, \end{aligned}$$

so $|T_\phi f - f|_\infty < \varepsilon/2$.

Let f_λ be the orthog. proj. onto $V(\lambda) \subseteq L^2(G)$.

Then the f_λ are orthog., so

$$(*) \qquad \sum_\lambda |f_\lambda|_2^2 = |f|_2^2 < \infty,$$

where

$$|f|_2 = \left(\int_G |f(x)|^2 dx \right)^{1/2}$$

For $q>0$, let

$$f'' = \sum_{|\lambda|>q} f_\lambda, \qquad f' = T_\phi(f'')$$

We have $f', f'' \in \bigoplus_{|\lambda|>q} V(\lambda) =: V_q$, which is f.d.

We claim that every $F \in V_q$ is a matrix coeff.

Since $V(\lambda)$ is right-mult. invariant, $\rho(g)F \in V_q \ \forall g \in G$

Then $W = \{\text{span } \rho(g)F \mid g \in G\} \leq V_q$ is a f.d. G-repn.

Let $L: W \to \mathbb{C}$, $L(\phi) := \phi(1)$. Then

$$L(\rho(g)F) = (\rho(g)F)(1) = F(g),$$

so F is a matrix coeff., proving the claim.

By $(*)$ we can choose q s.t.

$$\left| \sum_{0<|\lambda|<q} f_\lambda \right|_1 \leq \left| \sum_{0<|\lambda|<q} f_\lambda \right|_2 < \frac{\varepsilon}{2|\phi|_\infty}.$$

We have

$$T_\phi(f-f'') = T_\phi\left(\sum_{0\le|\lambda|<q} f_\lambda\right) = T_\phi\left(\sum_{0<|\lambda|<q} f_\lambda\right),$$

and so $|T_\phi(f-f'')|_\infty \le |\phi|_\infty \left|\sum_{0<|\lambda|<q} f_\lambda\right|_1 < \frac{\varepsilon}{2}$

Hence,

$$|f-f''|_\infty \le |f-T_\phi f|_\infty + |T_\phi f - T_\phi f''|_\infty < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

b) Let G be a compact gp. and $V \neq 0$ be a unitary repn. Let $v \in V \setminus \{0\}$. Let N be a nhbd. of 1 in G s.t. $|\pi(g)v - v| \le |v|/2$ $\forall g \in N$. Let $\phi \in C(G)$ have $\operatorname{supp}\phi \subseteq N$ and $\int_G \phi(g)\,dg = 1$.

Now,

$$\left\langle \int_G \phi(g)\pi(g)v\,dg,\ v\right\rangle = \langle v,v\rangle - \left\langle \int_N \phi(g)(v-\pi(g)v)\,dg,\ v\right\rangle$$

and

$$\left|\left\langle \int_N \phi(g)(v-\pi(g)v)\,dg,\ v\right\rangle\right| \le \int_N |v-\pi(g)v|\,dg\cdot|v| \le \frac{|v|^2}{2},$$

so $\int_G \phi(g)\pi(g)v\,dg \neq 0$.

Let $\varepsilon > 0$, and by (a), choose a matrix coeff. f s.t. $|f-\phi|_\infty < \varepsilon$. By unitarity,

$$\left| \int_G (f-\phi)(g)\,\pi(g)v\,dg \right| \le \varepsilon |v|,$$

So for small enough ε, $\int_G f(g)\pi(g)v\,dg \neq 0$.

Since f is a matrix coeff., so is $g \mapsto f(g^{-1})$ [Bump Prop 2.4]. So write $f(g^{-1}) = L(\rho(g)w)$ where (ρ, W) is a f.d. repn, $w \in W$, and $L: W \to \mathbb{C}$ is a linear functional.

Define the map $T: W \to V$

$$T(x) = \int_G L(\rho(g^{-1})x)\,\pi(g)v\,dg$$

T is G-equivariant by the usual change-of-variables argument. $T \neq 0$ since $T(w) = \int_G f(g)\pi(g)v\,dg \neq 0$.

Thus $\operatorname{im} T$ is a nonzero f.d. subrepn, so V has a f.d. irred subrepn. Then by Zorn's Lemma, the maximal direct sum of orthog. f.d. irreps is V itself.

c) The proof is similar to HW4 #4 (which is in the case of finite groups).

$\square$