Def 62: A <u>topological group</u> is a gp. that is also a (Hausdorff) top. sp. such that the multiplication and inverse maps are continuous. All maps (e.g. rep'ns) are assumed to be continuous.

Some important cases:

$$\begin{array}{ccc} \text{Compact Lie gps.} & \subseteq & \text{Lie gps.} \\ \cap\!\!| & & \cap\!\!| \\ \text{Compact gps.} & \subseteq & \underbrace{\text{locally compact}}_{\substack{\text{small-enough closed} \\ \text{nbhds. are compact}}} \text{ gps.} \end{array}$$

Def/Thm 63:

a) A <u>Borel measure</u> is a measure that is all open sets (and thus all <u>Borel sets</u>).

Any locally-compact Hausdorff space has a Borel measure.

b) A <u>left Haar measure</u> on a locally compact gp. G is a Borel measure μ_L that is invariant under left translation:

$$\mu_L(X) = \mu_L(gX) \text{ for all measurable } X, \text{ and } g \in G$$

A <u>right Haar measure</u> on G is a Borel measure μ_R that is invariant under right translation:

$$\mu_R(X) = \mu_R(gX) \text{ for all measurable } X, \text{ and } g \in G$$

a) Every locally-compact gp. has a left/right Haar measure, unique up to scalar multiple.

Pf: See sources in [Bump, Ch. 1]. $\square$

b) For compact groups, $\mu_L = \mu_R =: \mu$

Pf: [Bump Prop 1.1, 1.2] $\square$

We use this measure to integrate measurable functions on G:

$$\int_G f(g)\, dg := \int_G f(g)\, d\mu(g) \qquad \mu = \mu_L \text{ or } \mu_R$$

When G is compact, let us normalize μ s.t. $\mu(G) = 1$.

Recall:

$$L^2(G) = \left\{ f : G \to \mathbb{C} \,\middle|\, \int_G |f(g)|^2\, dg < \infty \right\}$$

Hermitian inner product: $\langle f, f' \rangle := \int_G f(g) \overline{f'(g)}\, dg$

G-action: $g \cdot f(g') := f(g^{-1} g')$

This is the most natural analogue of the reg. repn. (HW1 #4)

If G is a compact abelian gp., then all f.d. irreps of G are 1D (see HW4) and any $f \in L^2(G)$ has a "Fourier expansion"

$$f(g) = \sum_{\chi:\text{irred}} a_\chi \chi(g), \qquad a_\chi = \int_G \underbrace{f(g)\overline{\chi(g)}}_{L^2\text{-inner product}} dg$$

$$\text{and} \quad \int_G |f(g)|^2 dg = \sum_\chi |a_\chi|^2 \quad \text{(Plancherel formula)}$$

We want to show something similar for general compact gps.

Thm 64: Let G be a compact gp. Then every f.d. repn. V is the direct sum of irreps.

Pf: Use Weyl's averaging trick:

Let $\langle \cdot, \cdot \rangle$ denote any Hermitian inner prod. on V. Define

$$\langle v, w \rangle_G := \int_G \langle gv, gw \rangle dg$$

Now, $\langle \cdot, \cdot \rangle_G$ is also a Hermitian inner product: conj. symm., linear in first argument, pos. def.

It is also G-invariant:

$$\langle gv, gw \rangle_G = \int_G \langle hgv, hgw \rangle d\mu(h)$$

$$\overset{h'=hg}{=} \int_G \langle h'v, h'w \rangle \, d\mu(h') \qquad \left(\text{since } d\mu(h) = d\mu(gh)\right)$$

$$= \langle v, w \rangle_G$$

If $W \leq V$ is a subrepn, then $U := W^\perp$ (w.r.t. $\langle \cdot, \cdot \rangle_G$) is also a subrepn since if $u \in U, w \in W$, then

$$\langle gu, w \rangle_G = \langle u, \underbrace{g^{-1}w}_{\in W} \rangle_G = 0.$$

This every subrepn has an orthog. complement, so V is completely reducible. $\square$

(Compare to the pf. of Maschke's Thm in lecture 3)

Have Schur's Lemma, character orthogonality similarly to finite case

Def 6.5: A <u>matrix coefficient</u> for a repn (π, V) of a gp. G is a function of the form

$$\phi : G \to \mathbb{C}$$

$$\phi(g) := L(\pi(g)v)$$

where L is a linear functional $V \to \mathbb{C}$

Ex: If $v = e_i$ the ith basis vector and $L = e_j^*$ is the functional sending $v \in V$ to its jth component, then $\phi(g)$ is the (j,i)-entry of $\pi(g)$.

Def 66: A repn of a gp. G on a complex Hilbert space V (complete finite or ∞-dim'l Hermitian inner prod space) is <u>unitary</u> if $\langle gv, gw\rangle = \langle v, w\rangle \ \forall g \in G, \ v, w \in V$.

We have seen that f.d. repns of finite & compact gps. are <u>unitarizable</u> (able to be made unitary)

Thm 67 (Peter-Weyl Theorem): Let G be a compact group.

a) The matrix coeffs. of G dense in the set

$$C(G) = \{ f : G \to \mathbb{C} \mid f \text{ cont.} \}$$

of cont. funs. on G,

w.r.t. the L^∞-norm $|f - f'|_\infty = \sup_{g \in G} |f(g) - f'(g)|$

(and therefore in $L^2(G)$ since $C(G)$ is dense in $L^2(G)$).

b) Any unitary repn. of a compact gp. on a complex Hilbert space is completely reducible.

c) We have the (completed) direct sum decomposition

$$L^2(G) = \widehat{\bigoplus_{\pi}} V_\pi^{\oplus \dim V_\pi}$$

where the sum is over all unitary irreps. (π, V_π)

We'll use convolution of functions:

$$(f * f')(g) := \int_G f(gh^{-1}) f'(h)\, dh = \int_G f(h) f'(h^{-1}g)\, dh$$

which is always defined if $f, f' \in C(G)$.

If $\phi \in C(G)$, let T_ϕ be the $L^2(G)$-operator

$$(T_\phi f) = \phi * f$$

Let $V(\lambda) := V(\lambda, \phi) = \{ f \in L^2(G) \mid T_\phi f = \lambda f \}$
be the λ eigenspace.

T_ϕ is bounded, satisfying $|T_\phi f|_\infty \leq |\phi|_\infty |f|_1 \ \forall f \in L^1(G)$
and self-adjoint if $\phi(g^{-1}) = \overline{\phi(g)}$.
Using the Spectral Thm. for compact operators,
the $V(\lambda)$ are orthog., span $L^2(G)$, and if $\lambda \neq 0$,
$V(\lambda)$ is f.-d. In fact, if $q > 0$, $\sum_{|\lambda| > q} \dim V(\lambda) < \infty$

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In addition it is inv. under <u>right</u> mult. $(\rho(g) \cdot f(h) = f(hg))$
since if $T_\phi f = \lambda f$,

$$(T_\phi \rho(g) f)(x) = \int_G \phi(xh^{-1}) f(hg)\, dh = \int_G \phi(xgh^{-1}) f(h)\, dh =$$

$$\rho(g)(T_\phi f)(x) = (\lambda \rho(g) f)(x)$$

Pf sketch: a) Let $f \in C(G)$, $\varepsilon > 0$. We will show that there exists a matrix coeff. f' w/ $|f - f'|_\infty = 0$.

Since f is cont. on a compact set, $\exists$ a nbhd U of $1 \in G$ s.t.

$$|g \cdot f - f|_\infty < \varepsilon/2 \quad \forall g \in U$$

Let ϕ be a nonneg. fun supported on U s.t.

$$\int_G \phi(g)\,dg = 1 \text{ and } \phi(g^{-1}) = \overline{\phi(g)}.$$

Then if $h \in G$,

$$|T_\phi f(h) - f(h)| = \left| \int_G \left(\phi(g) f(g^{-1}h) - \phi(g) f(h)\right) dg \right|$$

$$\leq \int_U \phi(g)\,|f(g^{-1}h) - f(h)|\,dg$$

$$\leq \int_U \phi(g)\,|g \cdot f - f|_\infty\,dg$$

$$\leq \int_U \phi(g) \cdot \frac{\varepsilon}{2} \cdot dg = \frac{\varepsilon}{2},$$

so $|T_\phi f - f|_\infty < \varepsilon/2$.

Let f_λ be the orthog. proj. onto $V(\lambda) \subseteq L^2(G)$.

Then the f_λ are orthog., so

where

(*) $$\sum_\lambda |f_\lambda|_2^2 = |f|_2^2 < \infty,$$

$$|f|_2 = \left(\int_G |f(x)|^2 dx \right)^{1/2}$$

For $q>0$, let

$$f'' = \sum_{|\lambda|>q} f_\lambda, \qquad f' = T_\phi(f'')$$

We have $f', f'' \in \bigoplus_{|\lambda|>q} V(\lambda) =: V_q$, which is f.d.

We claim that every $F \in V_q$ is a matrix coeff.

Since $V(\lambda)$ is right-mult. invariant, $\rho(g)F \in V_q \ \forall g \in G$

Then $W = \{\text{span } \rho(g)F \mid g \in G\} \leq V_q$ is a f.d. G-repn.

Let $L: W \to \mathbb{C}$, $L(\phi) := \phi(1)$. Then

$$L(\rho(g)F) = (\rho(g)F)(1) = F(g),$$

so F is a matrix coeff., proving the claim.

By (*) we can choose q s.t.

$$\left| \sum_{0<|\lambda|<q} f_\lambda \right|_1 \leq \left| \sum_{0<|\lambda|<q} f_\lambda \right|_2 < \frac{\varepsilon}{2|\phi|_\infty}.$$

We have

$$T_\phi(f-f'') = T_\phi\left(\sum_{0\le|\lambda|<q} f_\lambda\right) = T_\phi\left(\sum_{0<|\lambda|<q} f_\lambda\right),$$

and so $|T_\phi(f-f'')|_\infty \le |\phi|_\infty \left|\sum_{0<|\lambda|<q} f_\lambda\right|_1 < \frac{\varepsilon}{2}$

Hence,

$$|f-f'|_\infty \le |f-T_\phi f|_\infty + |T_\phi f - T_\phi f''|_\infty < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

b), c) : next time