Recall: A <u>Lie group</u> is G is a gp. that is also a differentiable manifold s.t. the multiplication and inverse maps are smooth.

If G is a closed subgp. of $GL_n(\mathbb{C})$, it is a <u>matrix Lie gp.</u>

Examples of matrix Lie gps:

$GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $SL_n(\mathbb{R})$, $SL_n(\mathbb{C})$

$O(n)$, $SO(n)$, $U(n)$ $SU(n)$, $Sp_{2n}(\mathbb{R})$, $Sp_{2n}(\mathbb{C})$, $Sp(2n)$

The <u>matrix exponential</u> is the map

$$\mathrm{Mat}_n(\mathbb{C}) \to GL_n(\mathbb{C})$$

$$X \mapsto e^X := I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots \quad \text{(converges for all } X\text{)}$$

Def/Prop 59: Let G be a matrix Lie gp. The <u>Lie algebra</u> of G is the set $\mathfrak{g} = \mathrm{Lie}(G)$ of all matrices X s.t.

$$e^{tX} \in G \text{ for all } t \in \mathbb{R}. \leftarrow \text{not } \mathbb{C}!$$

$\mathrm{Lie}(G)$ is a Lie algebra in the sense of Def. 49

(for complex Lie gps.; for real Lie gps, $\mathrm{Lie}(G)$ is a <u>real Lie alg.</u>)

Remarks:

a) $\{e^{tX} \mid t \in \mathbb{R}\}$ is called a <u>one-parameter subgp.</u>

b) Since $\{e^{tX} \mid t \in \mathbb{R}\}$ is connected. and $e^{0X} = I$, e^X is an elt. of the identity component of G

c) $X = \frac{d}{dt} e^{tX} \Big|_{t=0}$, so $\mathrm{Lie}(G)$ is the "tangent space at the identity" of G.

Lie gp.	Lie alg.
$GL_n(\mathbb{C})$	$\mathfrak{gl}_n(\mathbb{C})$
$SL_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$
$U(n)$	$\mathfrak{u}(n) := \{X \in \mathfrak{gl}_n(\mathbb{C}) \mid \overline{X}^T + X = 0\}$ ← not a complex v.s.
$SU(n)$	$\mathfrak{su}(n) := \mathfrak{u}(n) \cap \mathfrak{sl}_n(\mathbb{C})$
$O(n)$	$\mathfrak{so}_n(\mathbb{C})$
$SO(n)$	$\mathfrak{so}_n(\mathbb{C})$
$Sp_n(\mathbb{C})$	$\mathfrak{sp}_n(\mathbb{C})$
$Sp(n)$	$\mathfrak{sp}_n(\mathbb{C}) \cap \mathfrak{u}(n)$

Class activity: Think about this relationship for $SL_n(\mathbb{C})$ and $\mathfrak{sl}_n(\mathbb{C})$

Def/Thm 60: Let G and H be Lie groups w/ Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. A <u>Lie gp. homomorphism</u> is a smooth/holomorphic gp. homom. (if over $\mathbb{C}$)

$$\phi : G \to H.$$

If $H = GL_n(\mathbb{C})$, then ϕ is a (complex) <u>Lie group repn.</u>

ϕ induces a Lie algebra homom. $\phi_* : \mathfrak{g} \to \mathfrak{h}$ given by

$$\phi_*(X) = \frac{d}{dt}\phi(e^{tX})\Big|_{t=0},$$

and conversely, $\phi(e^X) = e^{\phi_*(X)}$.

Furthermore, if G is simply connected, then for every Lie algebra homom. $\varphi : \mathfrak{g} \to \mathfrak{h}$ there is a unique Lie gp. homom. ϕ s.t. $\phi(e^X) = e^{\varphi(X)}$.

Ex: Let $Ad : G \to Aut(G)$

$$A \mapsto Ad_A \quad \text{where } Ad_A(B) = ABA^{-1}$$

Then Ad_* is the adjoint rep'n $ad : \mathfrak{g} \mapsto \mathfrak{gl}(\mathfrak{g})$ since $e^{ad_X} = Ad_{e^X}$ (HW4?)

Prop 61: Let G be a conn. Lie gp w/ Lie alg. $\mathfrak{g}$.
Let $\Pi, \Pi' : G \to GL(V)$ be Lie gp. repns
and let $\pi, \pi' : \mathfrak{g} \to \mathfrak{gl}(V)$ be the corresp.
Lie alg. repns. Then

a) $\Pi(e^X) = e^{\pi(X)}$

b) $\pi(X) = \frac{d}{dt} \Pi(e^{tX})\Big|_{t=0}$

c) Π is irred $\Leftrightarrow$ π is irred.

d) $\Pi \cong \Pi' \Leftrightarrow \pi \cong \pi'$

e) If $X \in \mathfrak{g}$, $A \in G$, then $AXA^{-1} \in \mathfrak{g}$ and

$$\pi(AXA^{-1}) = \Pi(A)\pi(X)\Pi(A)^{-1}$$

Therefore, if G: simply connected, the irreps. of G and $\mathfrak{g}$ are in correspondence

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Let $G = SL_n(\mathbb{C})$: simply connected

$\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) = \text{Lie}(G)$

For $\alpha \in \Phi$, $e_\alpha, f_\alpha, h_\alpha \in \mathfrak{sl}_2 \subseteq \mathfrak{sl}_3$

Then $SL_3(\mathbb{C})$ is gen'd by:

$E_\alpha(t) := \exp(te_\alpha)$, $F_\alpha(t) := \exp(tf_\alpha)$, $H_\alpha(t) := \exp(th_\alpha)$

$$E_{\alpha_1}(t) = \begin{bmatrix} 1 & t & \\ & 1 & \\ & & 1 \end{bmatrix} \quad F_{\alpha_1}(t) = \begin{bmatrix} 1 & & \\ t & 1 & \\ & & 1 \end{bmatrix} \quad H_{\alpha_1}(t) = \begin{bmatrix} e^t & & \\ & e^{-t} & \\ & & 1 \end{bmatrix}$$

$$E_{\alpha_2}(t) = \begin{bmatrix} 1 & & \\ & 1 & t \\ & & 1 \end{bmatrix} \quad F_{\alpha_2}(t) = \begin{bmatrix} 1 & & \\ & 1 & \\ & t & 1 \end{bmatrix} \quad H_{\alpha_2}(t) = \begin{bmatrix} 1 & & \\ & e^t & \\ & & e^{-t} \end{bmatrix}$$

$$E_{\rho}(t) = \begin{bmatrix} 1 & & t \\ & 1 & \\ & & 1 \end{bmatrix} \quad F_{\rho}(t) = \begin{bmatrix} 1 & & \\ & 1 & \\ t & & 1 \end{bmatrix} \quad H_{\rho}(t) = \begin{bmatrix} e^t & & \\ & 1 & \\ & & e^{-t} \end{bmatrix}$$

Let V^λ be the highest wt. repn. assoc. to λ, and let μ be a weight w/ wt. vector v_μ

Then,

$$\Pi(e^{th})v_\mu = e^{\pi(th)} = \sum_{m \geq 0} \frac{\pi(th)^m}{m!} v_\mu$$

$$= \sum_{m \geq 0} \frac{\mu(th)^m}{m!} v_\mu = \sum_{m \geq 0} \frac{t^m \mu(h)^m}{m!} v_\mu$$

$$= e^{t\mu(h)} v_\mu \qquad \text{still an eigenvector}$$

However,

$$\Pi(e^{te_\alpha})v_\mu = e^{\pi(te_\alpha)}v_\mu = \sum_{m>0} \frac{\pi(te_\alpha)^m}{m!} v_\mu$$

$$= \sum_{m>0} \frac{t^m}{m!} \underbrace{\pi(e_\alpha)^m v_\mu}_{\in V_{\mu+m\alpha}}$$

not cleanly a raising operator

Next time: Compact groups and complete reducibility