Last time:

Theorem of the highest weight:

Let $\mathfrak{g}$ be a (f.d. complex) semisimple Lie algebra.

a) Every $\mathfrak{g}$-irrep. π is highest-weight.

b) π is determined by its highest weight

c) The highest weights of $\mathfrak{g}$-irreps are precisely the dominant integral elements.

Properties of highest weight representations:

Let (π^λ, V^λ) denote the $\mathfrak{g}$-irrep. w/ highest. wt. λ

- The highest weight has multiplicity 1
- The action of the Weyl gp. sends weight spaces to wt. spaces of the same multiplicity.
- $\mu \in \mathfrak{h}^*$ is a wt. for V^λ iff μ is contained in the convex hull of $W\lambda$ and $\lambda - \mu$ is an integer linear combination of the simple roots

- Weyl character formula: Let

$$\mathrm{ch}_\lambda(\underset{\in\mathfrak{h}}{H}) := \mathrm{Tr}\left(e^{\pi_\lambda(H)}\right) = \sum_\mu m_\mu e^{\mu(H)} \in \mathbb{C}$$

mult. of the wt. μ

Then

$$\mathrm{ch}_\lambda(H) = \frac{\sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)(H)}}{\underbrace{\sum_{w\in W} (-1)^{\ell(w)} e^{w(\rho)(H)}}_{} = \prod_{\alpha\in\Phi^+}\left(e^{\alpha(H)/2} - e^{-\alpha(H)/2}\right)}$$

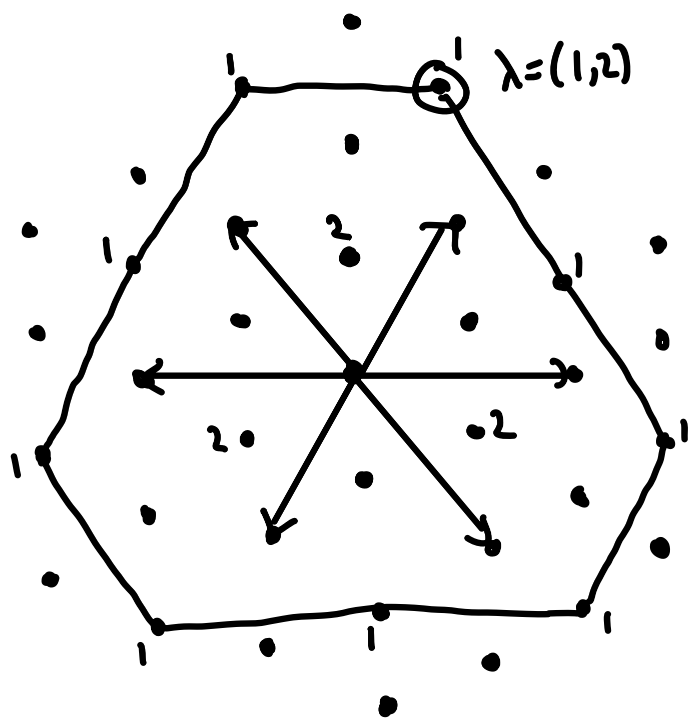
where $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$

(
- Can extend to $\mathfrak{g}$ by diagonalizability/continuity
- With exponentiation, $e^\mu \cdot e^{\mu'} = e^{\mu+\mu'}$ and $\mathrm{ch}_\lambda \cdot \mathrm{ch}_{\lambda'} = \mathrm{ch}_{\lambda+\lambda'} + \cdots$
)

- Weyl dimension formula:

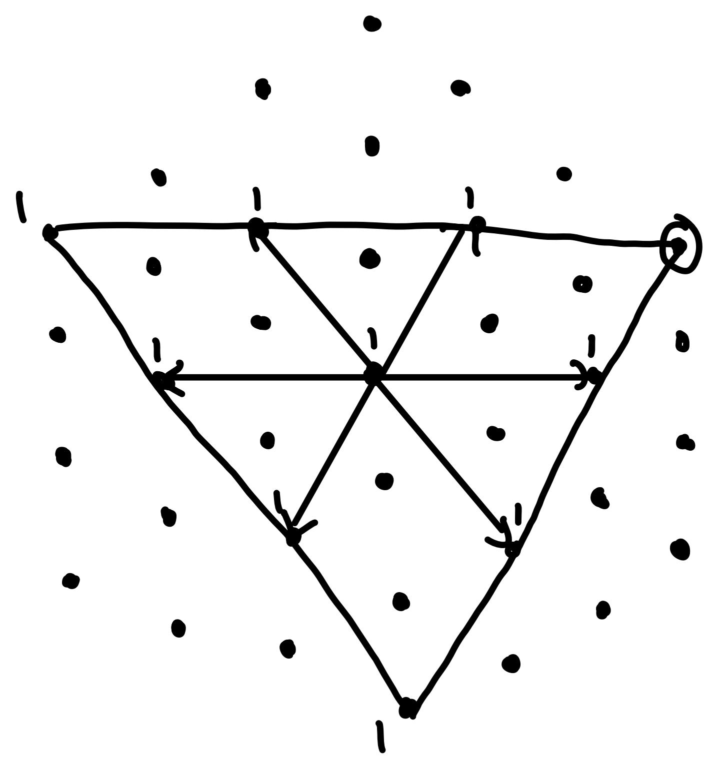
$$\dim V^\lambda = \prod_{\alpha\in\Phi^+} \frac{\langle \lambda+\rho, \alpha\rangle}{\langle \rho, \alpha\rangle}$$

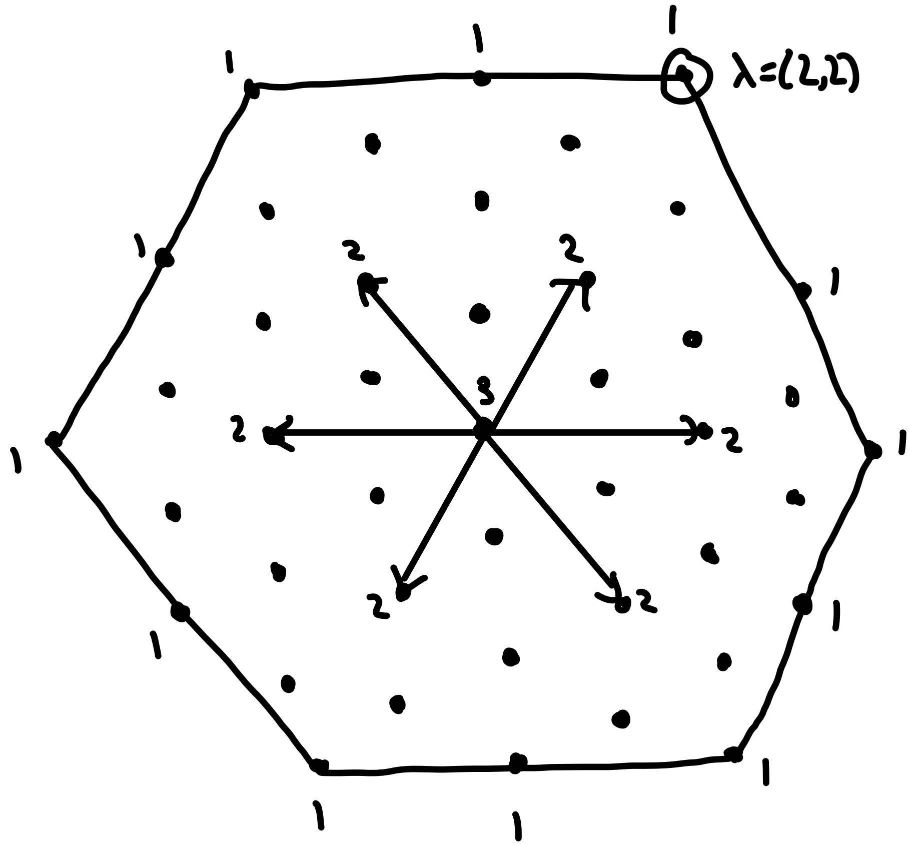
Examples for $\mathfrak{g} = \mathfrak{sl}_3$:



Class activity:

compute $\dim V^\lambda$




λ=(2,2)
1
2
3

Lie groups [Hall, Ch. 1] [Bump, Ch. 5]

Def 58: A <u>Lie group</u> is G is a gp. that is also a differentiable manifold s.t. the multiplication and inverse maps

$$\begin{array}{ccc} G\times G \to G & \text{and} & G \to G \\ (g,h) \mapsto gh & & g \mapsto g^{-1} \end{array}$$

are smooth.

If G is a closed subgp. of $GL_n(\mathbb{C})$, it is a <u>matrix Lie gp.</u>

Examples of matrix Lie gps:

$GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $SL_n(\mathbb{R})$, $SL_n(\mathbb{C})$

Orthogonal group: $O(n) = \{g \in GL_n(\mathbb{C}) \mid gg^T = I\}$

Special orthogonal gp: $SO(n) = O(n) \cap SL_n(\mathbb{C})$

Unitary gp: $U(n) = \{g \in GL_n(\mathbb{C}) \mid g\overline{g^T} = I\}$

Special unitary gp.: $SU(n) = U(n) \cap SL_n(\mathbb{C})$

Symplectic gp.: $Sp_{2n}(F) = \{g \in GL_{2n}(F) \mid gJg^T = J\}$, $J = \left[\begin{array}{c|c} 0 & \begin{smallmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{smallmatrix} \\ \hline \begin{smallmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{smallmatrix} & 0 \end{array}\right]$

Compact symplectic gp: $Sp(2n) = Sp_{2n}(\mathbb{C}) \cap U(2n)$

Lie gp.	Compact?	Connected?	Simply connected?
$GL_n(\mathbb{R})$	×	×	n/a
$GL_n(\mathbb{C})$	×	✓	×
$SL_n(\mathbb{R})$	×	✓	×
$SL_n(\mathbb{C})$	×	✓	✓
$O(n)$	✓	×	n/a
$SO(n)$	✓	✓	×
$U(n)$	✓	✓	×
$SU(n)$	✓	✓	✓
$Sp_{2n}(\mathbb{R})$	×	✓	×
$Sp_{2n}(\mathbb{C})$	×	✓	✓
$Sp(2n)$	✓	✓	✓

The <u>matrix exponential</u> is the map

$$\mathrm{Mat}_n(\mathbb{C}) \to GL_n(\mathbb{C})$$

$$X \mapsto e^X := I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots \quad \text{(converges for all } X\text{)}$$

Def 59: Let G be a matrix Lie gp. The <u>Lie algebra</u> of G is the set $\mathrm{Lie}(G)$ of all matrices X s.t.

$$e^{tX} \in G \text{ for all } t \in \mathbb{R}. \longleftarrow \text{not } \mathbb{C}!$$

Remark: Note that $X = \frac{d}{dt} e^{tX} \big|_{t=0}$

Lie gp.	Lie alg.
$GL_n(\mathbb{C})$	$\mathfrak{gl}_n(\mathbb{C})$
$SL_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C})$
$U(n)$	$\mathfrak{u}(n) := \{ X \in \mathfrak{gl}_n(\mathbb{C}) \mid \overline{X}^T + X = 0 \}$
$SU(n)$	$\mathfrak{su}(n) := \mathfrak{u}(n) \cap \mathfrak{sl}_n(\mathbb{C})$
$O(n)$	$\mathfrak{so}_n(\mathbb{C})$
$SO(n)$	$\mathfrak{so}_n(\mathbb{C})$
$Sp_n(\mathbb{C})$	$\mathfrak{sp}_n(\mathbb{C})$
$Sp(n)$	$\mathfrak{sp}_n(\mathbb{C}) \cap \mathfrak{u}(n)$

Next time: repns of Lie gps.