Recall:

Killing-Cartan classification:

Complex semisimple Lie algebras are in correspondence w/ root systems (and therefore Dynkin diagrams).

Root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

where $\mathfrak{h}$ is a Cartan subalgebra and

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} \mid \mathrm{ad}_H(X) = \alpha(H) X \ \forall H \in \mathfrak{h} \}$$

is the root space corresp. to α.

Note that this isn't just a structure theorem for $\mathfrak{g}$, but also for its adjoint rep'n

$$\mathrm{ad}_{\mathfrak{g}} : X \mapsto \mathrm{ad}_X \in \mathfrak{gl}(\mathfrak{g})$$

We'll see today that other rep'ns have a similar structure!

Def 55: Let $\mathfrak{g}$ be a semisimple Lie algebra w/ Cartan subalg. $\mathfrak{h}$. Let $\pi: \mathfrak{g} \to \mathfrak{gl}(V)$ be a f.d. repn. of $\mathfrak{g}$.

A <u>weight</u> for π is a linear functional $\lambda: \mathfrak{h} \to \mathbb{C} \in \underbrace{\mathfrak{h}^*}_{\text{dual space}}$ s.t. $\exists v \in V \setminus \{0\}$ with

$$\pi(H)v = \lambda(H)v \quad \forall H \in \mathfrak{h} \qquad (*)$$

The <u>weight space</u> corresp. to λ is the subspace V_λ of all $v \in V$ satisfying $(*)$. $v \in V_\lambda \setminus \{0\}$ is a <u>weight vector</u> for λ.

The <u>multiplicity</u> of the weight λ is $\dim V_\lambda$.

Ex: If π is the adjoint repn, the weights of π are $\Phi \sqcup \{0\}$.

- $\alpha \in \Phi$ has mult. 1, $V_\alpha = \mathfrak{g}_\alpha$
- $\lambda = 0$ has mult. n, $V_0 = \mathfrak{h}$

"roots are weights of the adjoint representation"

Recall the pairing $\langle \beta, \alpha \rangle := 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$ on $\mathfrak{h}^*$

Def 56: Call an element $\lambda \in \mathfrak{h}^*$ integral if $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$.

Call $\lambda \in \mathfrak{h}^*$ dominant if $\langle \lambda, \alpha \rangle \geq 0 \ \forall \alpha \in \Phi^+$.

If $\lambda, \mu \in \mathfrak{h}^*$, then λ is higher than μ if $\lambda - \mu$ is a nonnegative linear comb. of simple roots.

A weight λ of a $\mathfrak{g}$-repn π is a highest weight if λ is higher than every other weight of π, and π is called a highest-weight representation.

Thm 57 (Theorem of the highest weight):

Let $\mathfrak{g}$ be a (f.d. complex) semisimple Lie algebra.

a) Every $\mathfrak{g}$-irrep. π is highest-weight.

b) π is determined by its highest weight

c) The highest weights of $\mathfrak{g}$-irreps are precisely the dominant integral elements.

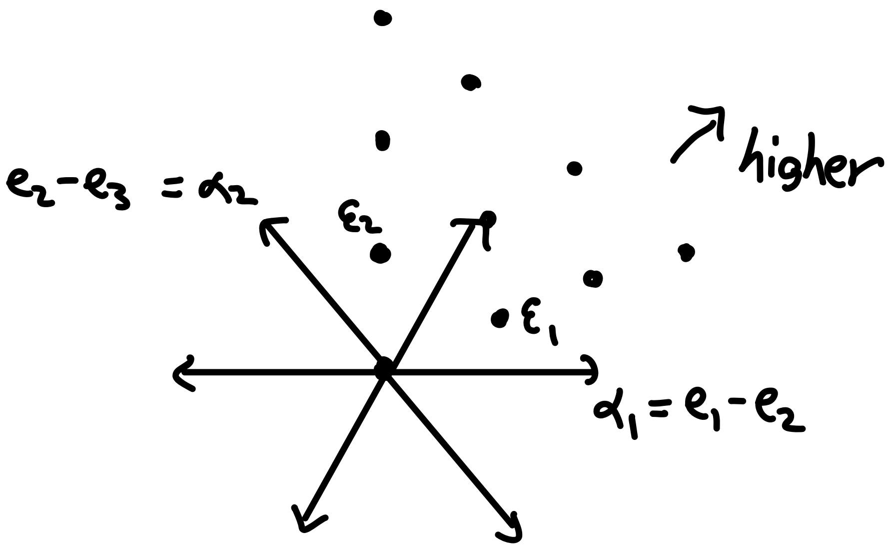
Already proved this for $\mathfrak{g} = \mathfrak{sl}_2$.

Dominant integral elts. $\leftrightarrow$ nonneg. integers.

For concreteness, we'll restrict to $\mathfrak{g} = \mathfrak{sl}_3$.

Dominant integral weights:

α_1, α_2: simple roots



They are the nonneg. int. combinations of the <u>fundamental weights</u>: $\varepsilon_1 = e_1$

$$\varepsilon_2 = e_1 + e_2$$

Satisfy $\langle \varepsilon_i, \alpha_j \rangle = \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$

[Notice that the highest root (highest wt. of adjoint repn) is the sum of the fundamental weights)]

$$h_1 = \begin{bmatrix} 1 & & \\ & -1 & \\ & & \end{bmatrix}, \quad e_1 = \begin{bmatrix} & 1 & \\ & & \\ & & \end{bmatrix}, \quad f_1 = \begin{bmatrix} & & \\ 1 & & \\ & & \end{bmatrix}$$

$\mathfrak{h}$

$$h_2 = \begin{bmatrix} & & \\ & 1 & \\ & & -1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} & & \\ & & 1 \\ & & \end{bmatrix}, \quad f_2 = \begin{bmatrix} & & \\ & & \\ & 1 & \end{bmatrix}$$

Weight: simultaneous h_1, h_2-evector

ε_1: h_1-evalue: 1, h_2-evalue: 0

ε_2: h_1-evalue: 0, h_2-evalue: 1

Write $(a,b) := a\varepsilon_1 + b\varepsilon_2$

Thm 57 pf. outline: [Hall, §5.4] (Should we spend a lecture on this proof?)

Part 1: Every $\mathfrak{g}$-irrep. (π, V) has at least one weight

Part 2: Let λ be a weight for π, $\alpha \in \Phi$.

Then, if $v \in V_\lambda$, $e_\alpha v \in V_{\lambda+\alpha}$ and $f_\alpha v \in V_{\lambda-\alpha}$

Part 3: V is the direct sum of its weight spaces

Part 4: Call a $\mathfrak{g}$-repn (π, V) <u>highest-weight cyclic</u> w/ weight μ if $\exists$ $v \in V_\mu \setminus \{0\}$ s.t $\pi(\mathfrak{g})v = V$ and $\pi(e_1)v = \pi(e_2)v = 0$. In such a repn, μ is the highest weight of π and $\dim V_\mu = 1$.

Part 5: Every $\mathfrak{g}$-irrep. is highest-weight cyclic, w/ a unique highest wt.

Part 6: Every highest-weight cyclic $\mathfrak{g}$-repn. is irred.

Part 7: Two $\mathfrak{g}$-irreps. w/ the same highest wt. are equivalent

Part 8: All highest weights are dominant integral

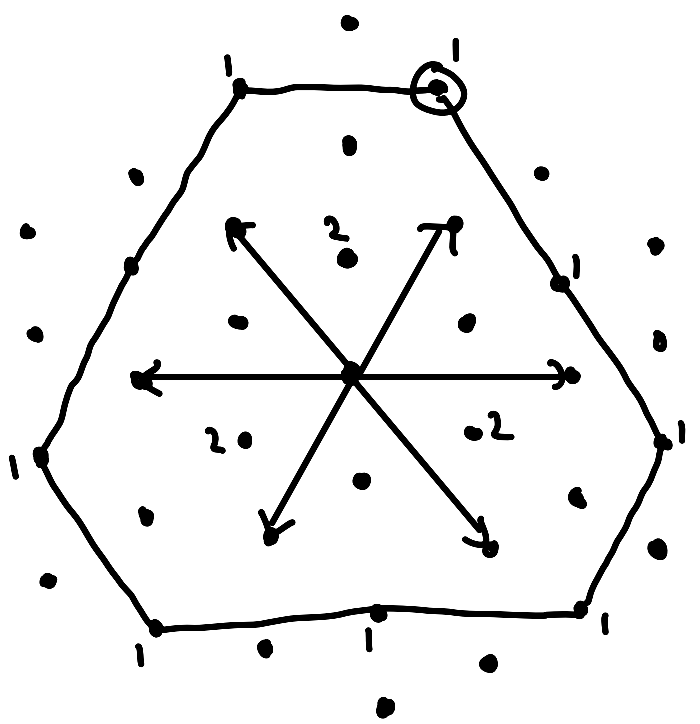
Part 9: All dominant integral elts. are highest weights

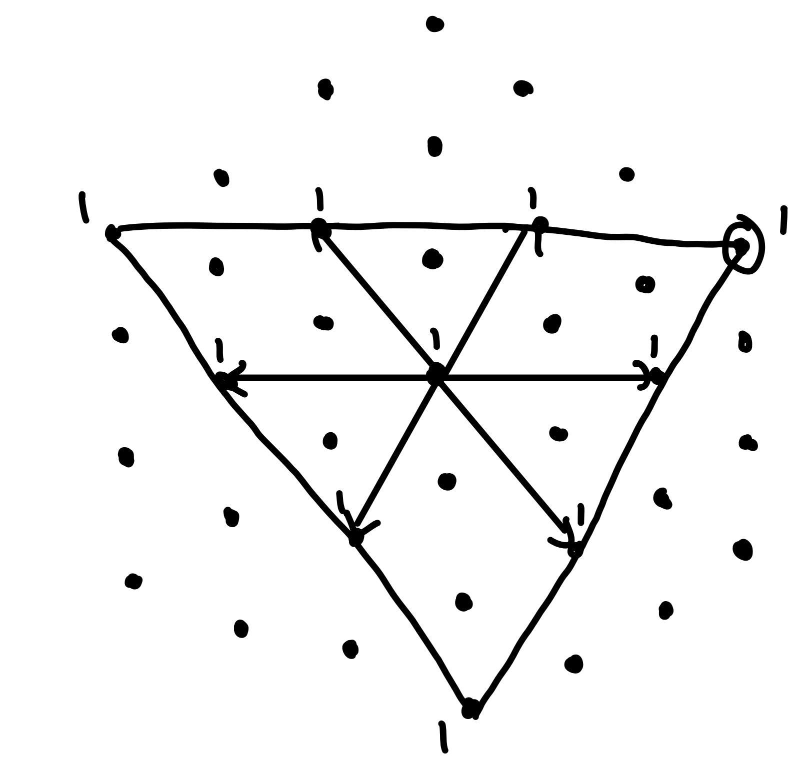
(If time) Properties:

Let V^λ denote the $\mathfrak{g}$-irrep. w/ highest. wt. λ

- The action of the Weyl gp. sends weight spaces to wt. spaces of the same multiplicity.
- $\mu \in \mathfrak{h}^*$ is a wt. for V^λ iff μ is contained in the convex hull of $W\lambda$ and $\lambda - \mu$ is an integer linear combination of the simple roots

Examples for $\mathfrak{g} = \mathfrak{sl}_3$:

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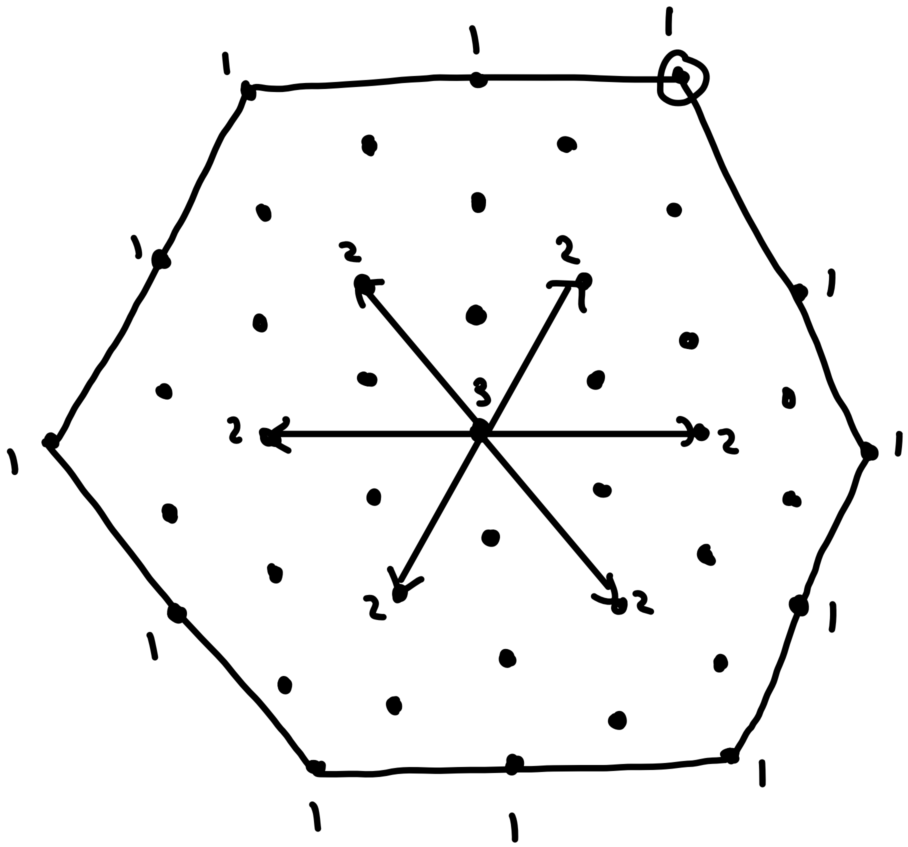
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